Discrete Graph Operators for Manifold-Valued Data

Habilitation colloquium at Department of Mathematics

Dr. Daniel Tenbrinck

Friedrich-Alexander-Universität Erlangen-Nürnberg

4th November, 2020

Me in a Nutshell

Short Academic CV

2009: Diploma in computer science / mathematics from WWU Münster

2013: PhD degree at WWU Münster with Prof. Jiang & Prof. Burger
"Variational methods for Medical Ultrasound Imaging"

2014: Postdoc at ENSICAEN (France) with Prof. Brun & Prof. Elmoataz

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Research Interests:

- Imaging
- Graph-based Methods
- Data Science

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Research Interests:

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- Data Science

Hobbys:

- my three little kids
- baking sourdough bread (if flour is available!)
- computers and technology

Outline

Introduction

- Finite Weighted Graphs for Data Processing
- Manifold-Valued Data Processing

Methods

- First-Order Difference Operators for Manifold-Valued Functions
- Graph p-Laplacian for Manifold-Valued Functions

Applications

- Synthetic Manifold-Valued Data
- Real Manifold-Valued Data

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How it all began...

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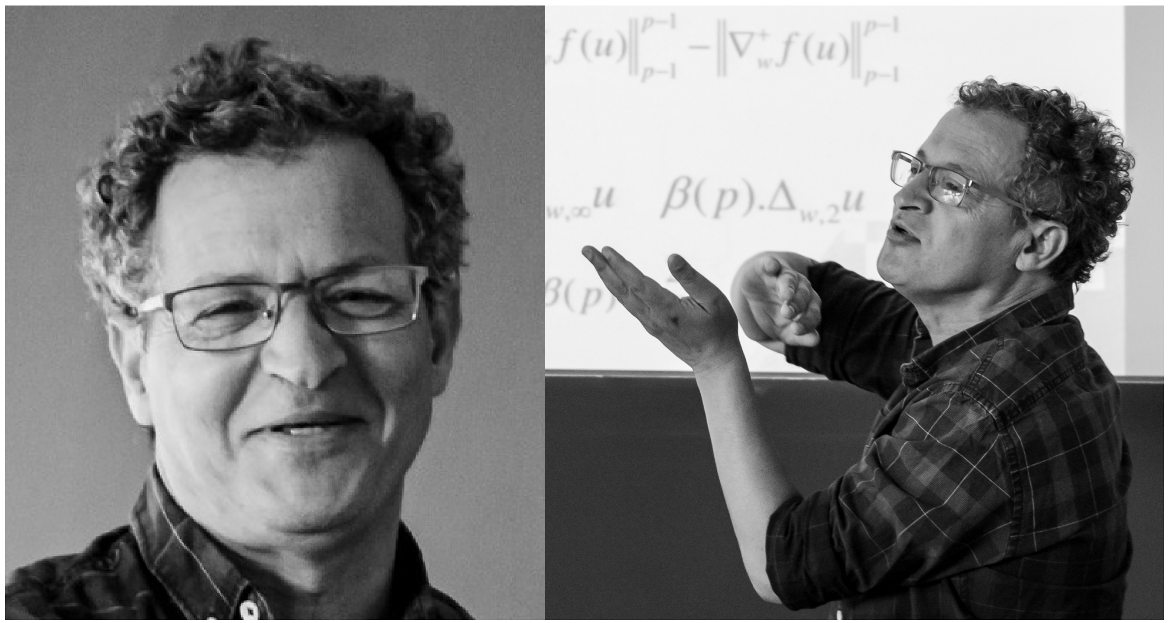

Prof. Abderrahim Elmoataz

- Initial work with A. Elmoataz, F. Lozes, and M. Toutain
- Ongoing collaboration on papers and grant proposals

Finite weighted graphs

A finite weighted graph $G = (V, E, w)$ consists of:

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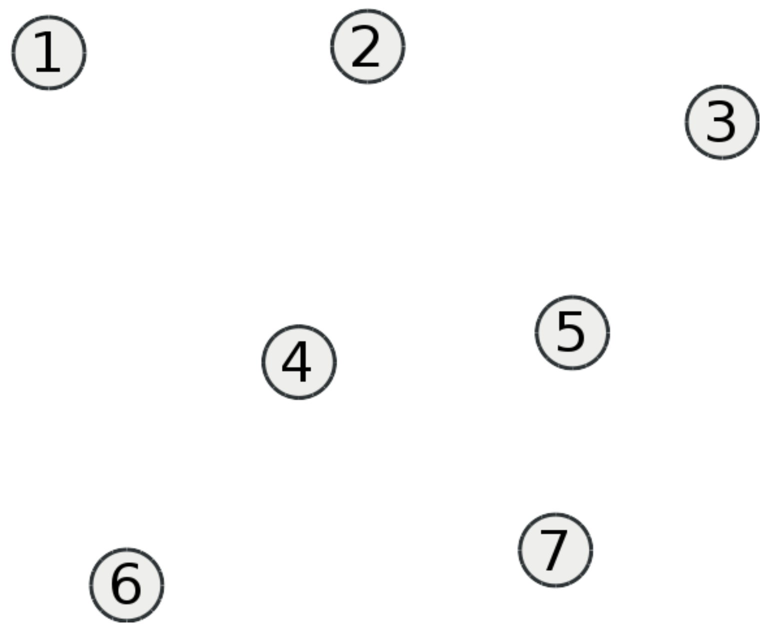
- a finite set of vertices $V = (v_1, \ldots, v_n)$



Finite weighted graphs

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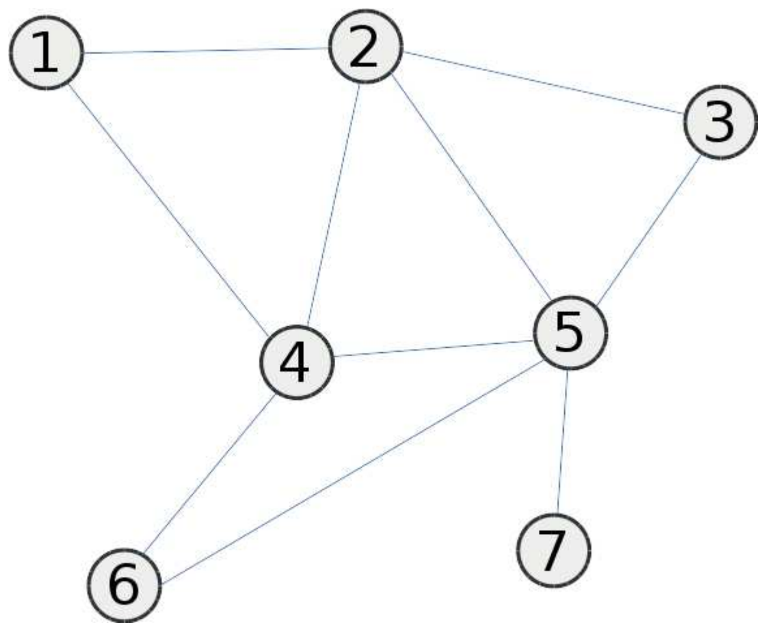

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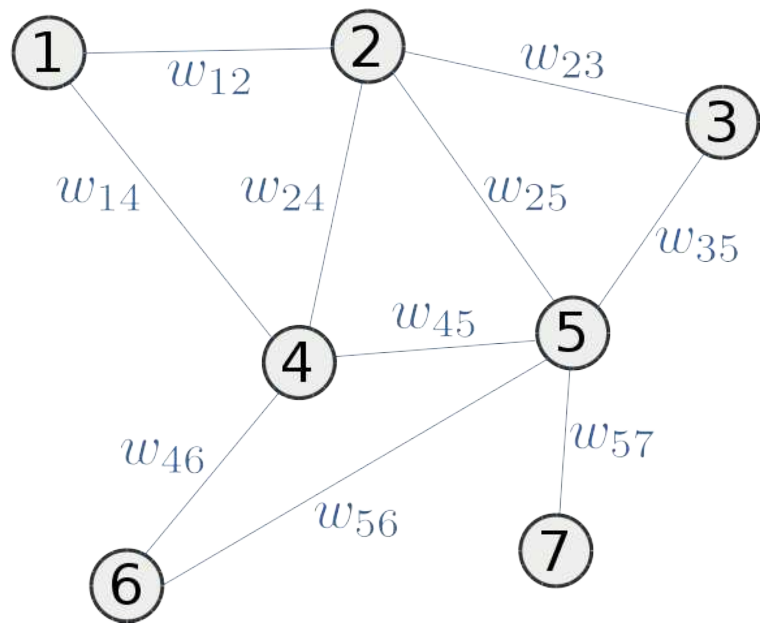
- a weight function $w\colon E \rightarrow [0, 1]$ with: $w(u, v) > 0 \Leftrightarrow (u, v) \in E$



Finite weighted graphs for modeling discrete data

Question: How can we apply graphs for image processing?

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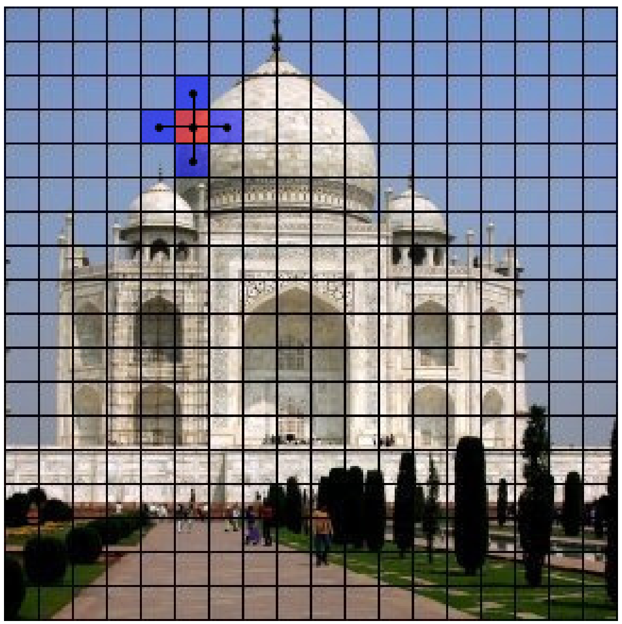

Local neighborhood of a pixel

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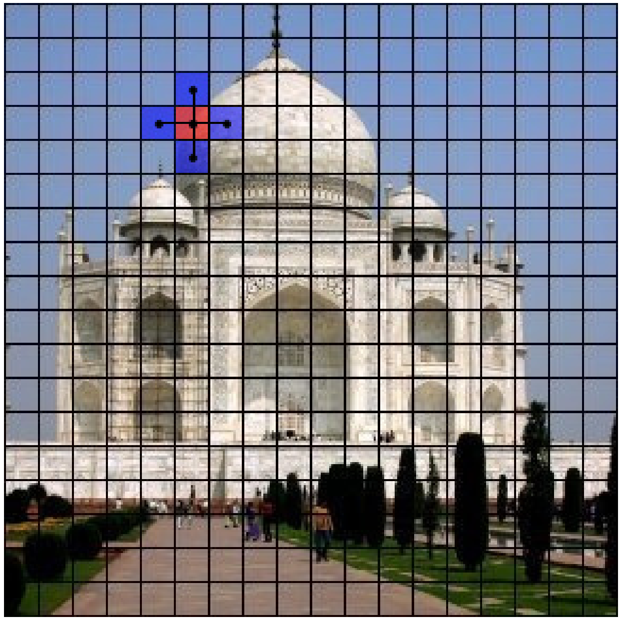

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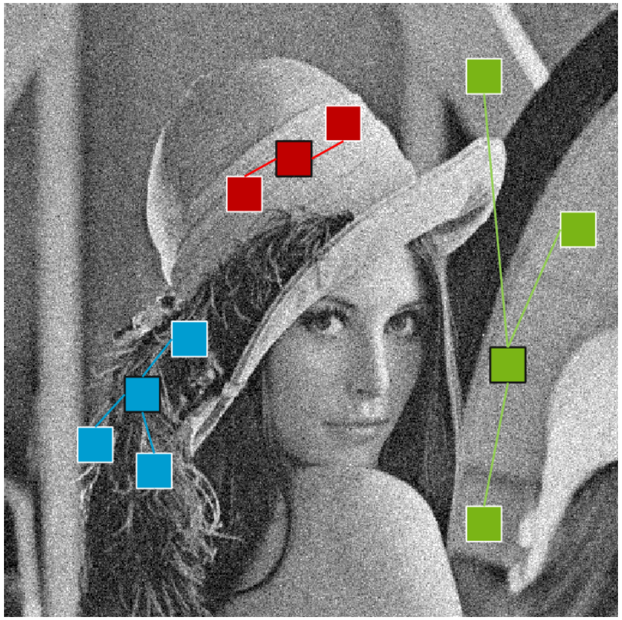

Nonlocal neighborhood of a pixel

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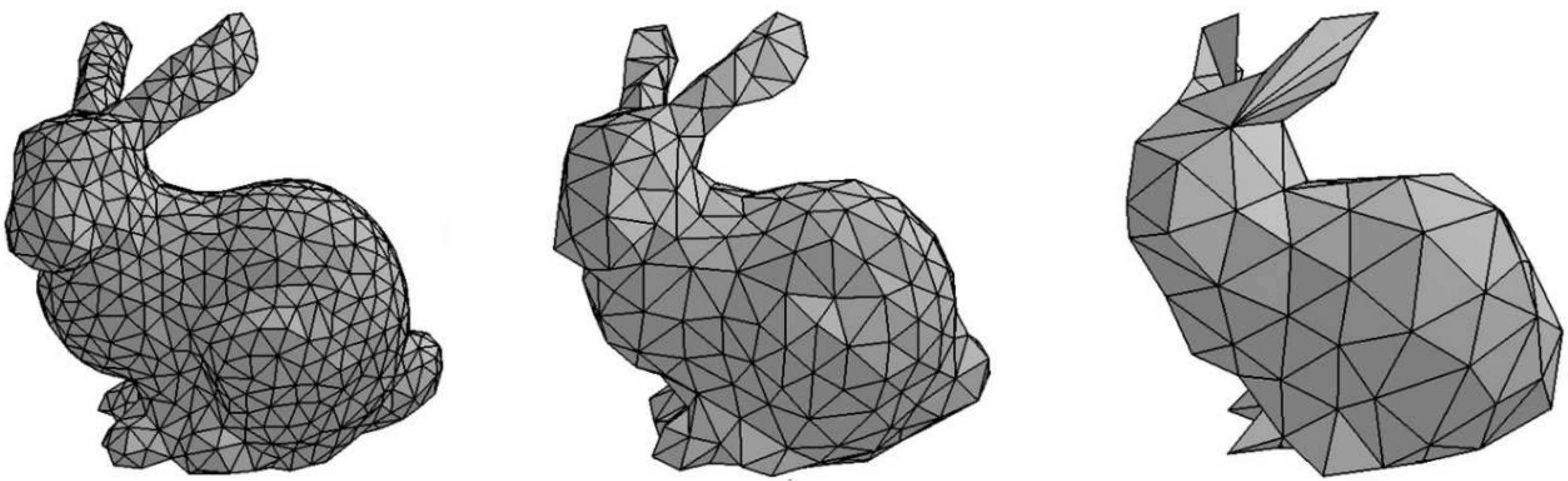

Polygon mesh approximation of a 3D surface.

Image courtesy: Gabriel Peyré

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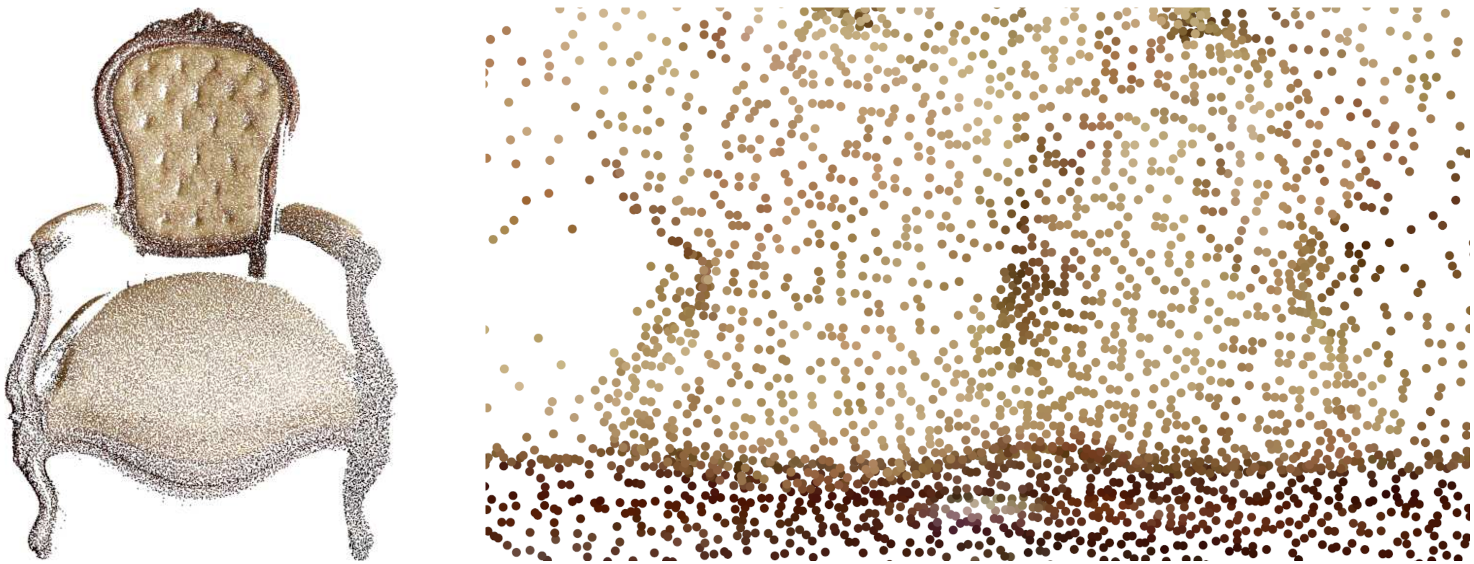

Colored 3D point cloud data of a scanned chair.

Image courtesy: François Lozes

Finite weighted graphs for modeling discrete data

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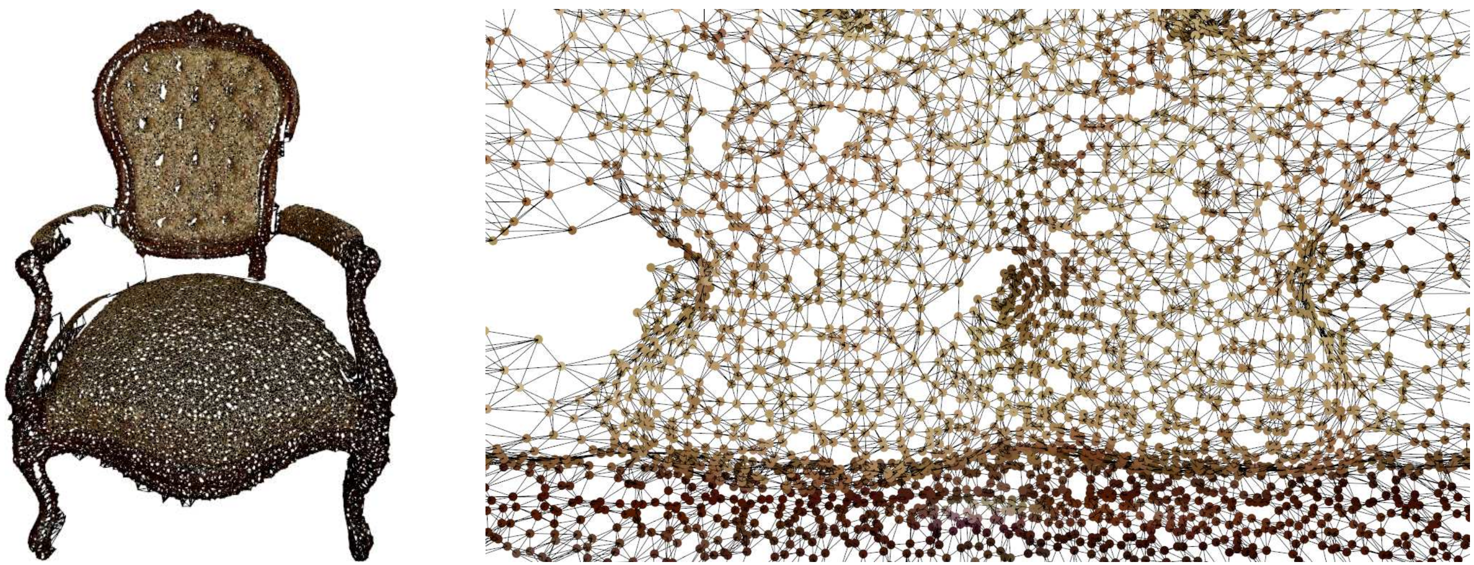

Graph construction on a 3D point cloud

Image courtesy: François Lozes

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Question: How can we apply graphs for machine learning?

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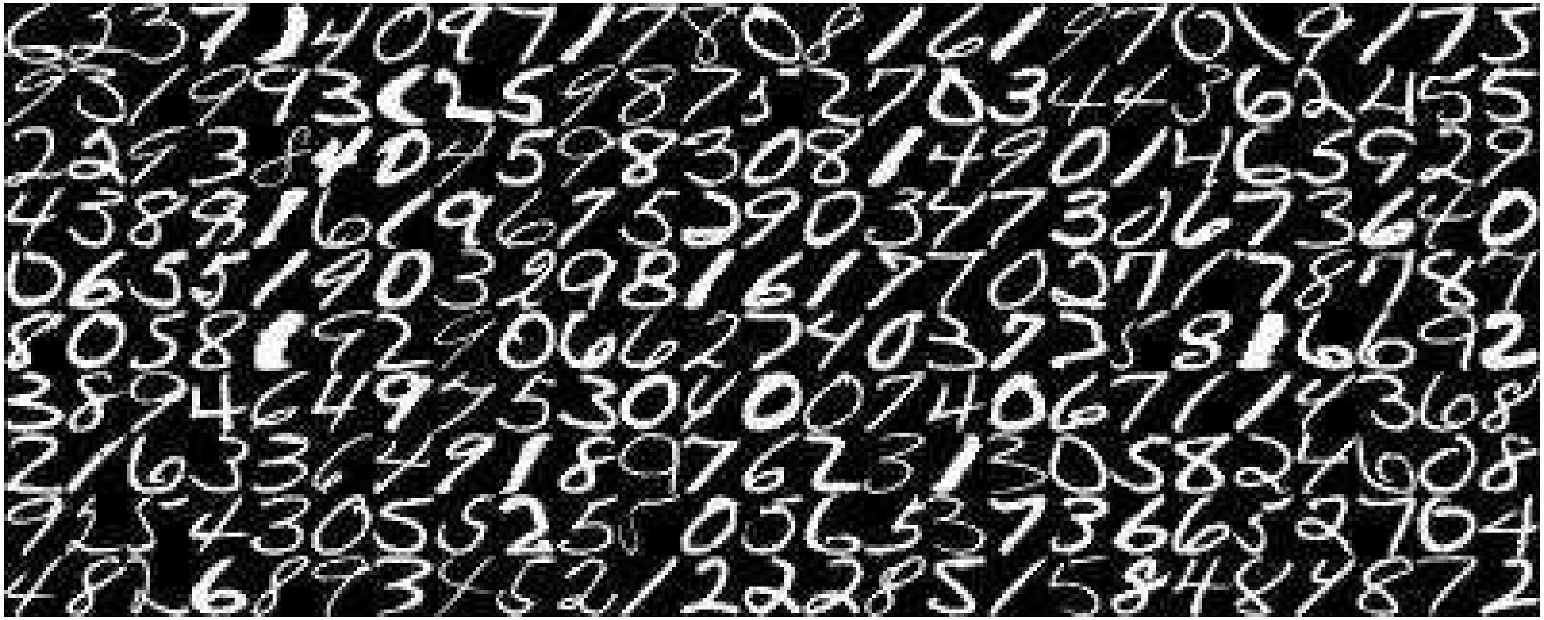

Subset of handwritten digits from USPS database [1]

[1] J. Hull: A database for handwritten text recognition research. IEEE Transactions on Pattern Analysis and Machine Intelligence 16(5). (1994)

Finite weighted graphs for modeling discrete data

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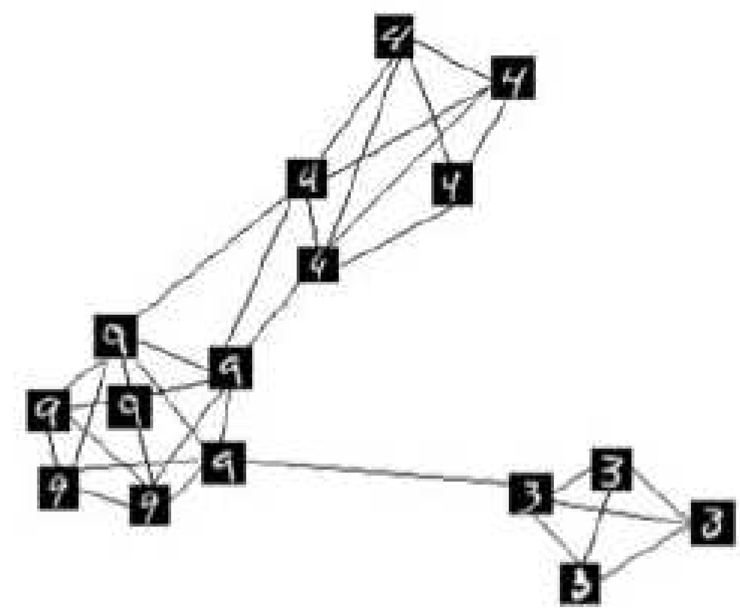

Graph construction using suitable similarity features.

Finite weighted graphs for modeling discrete data

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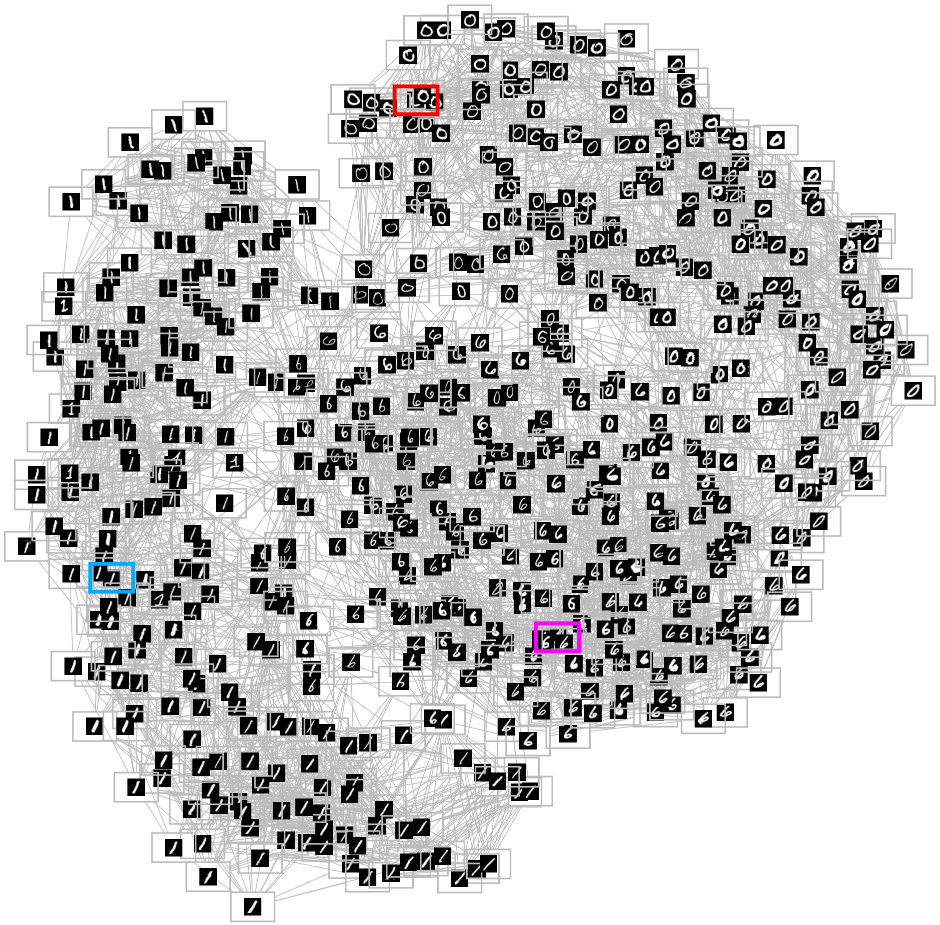

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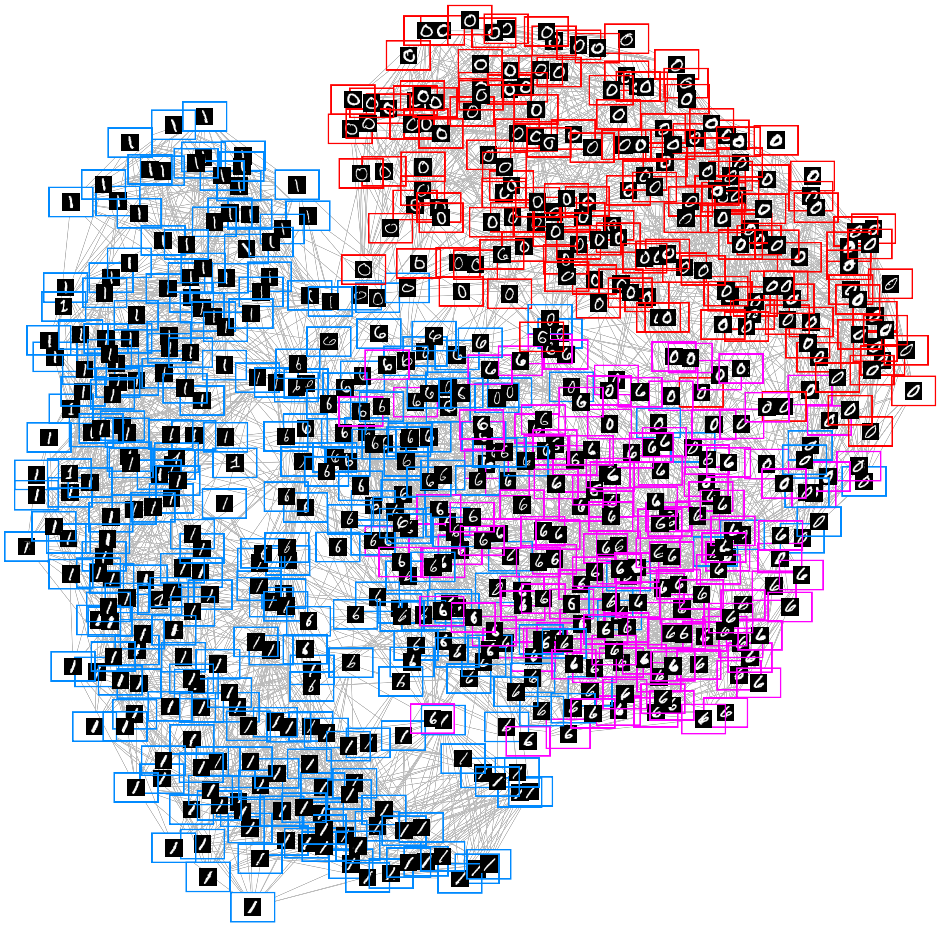
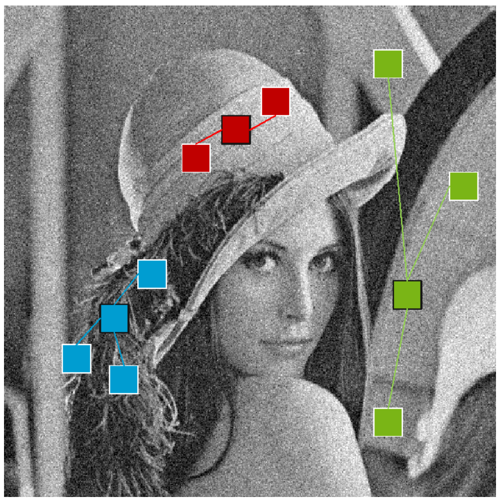
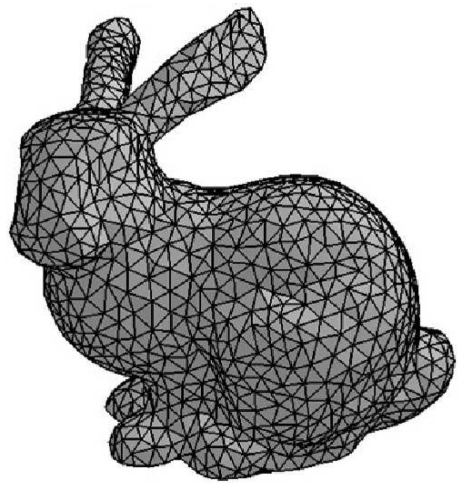
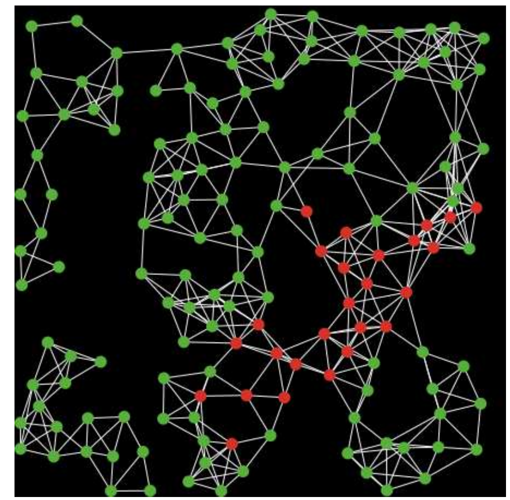


The graph framework

Given: Application data represented by a vertex function $f : V \to \mathbb{R}^m$.

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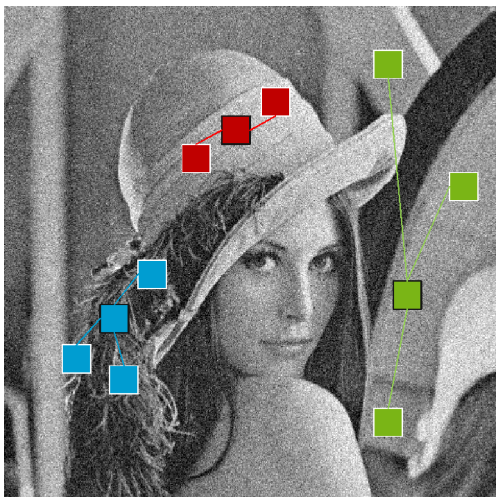
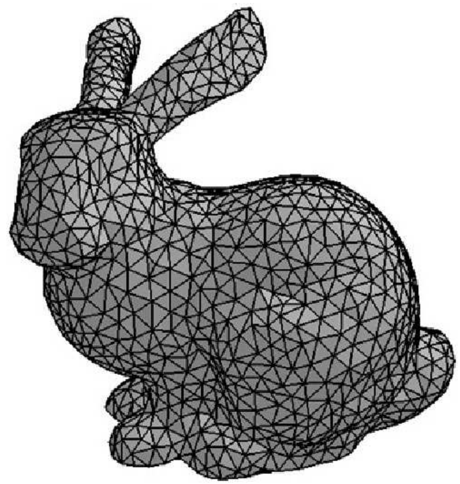
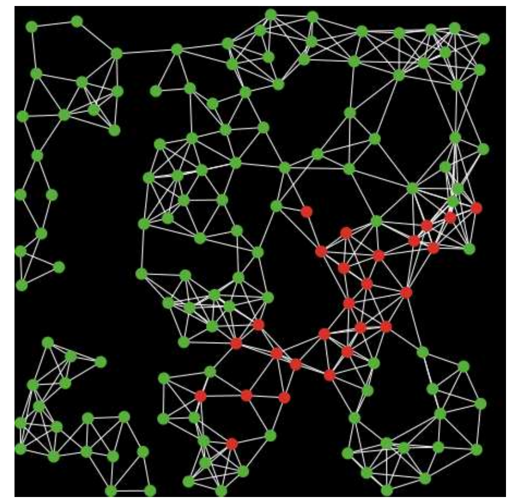

Abderrahim Elmoataz:

"Le monde - c'est tout un graph."

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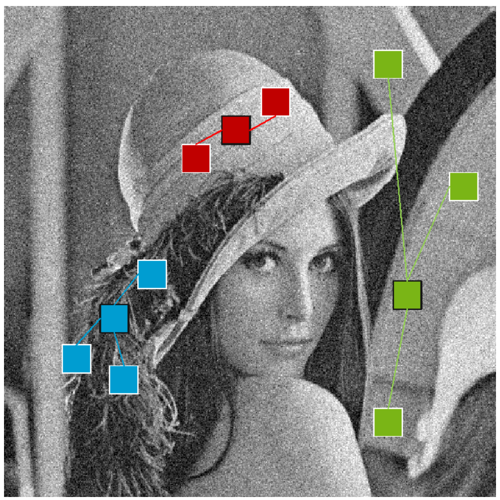

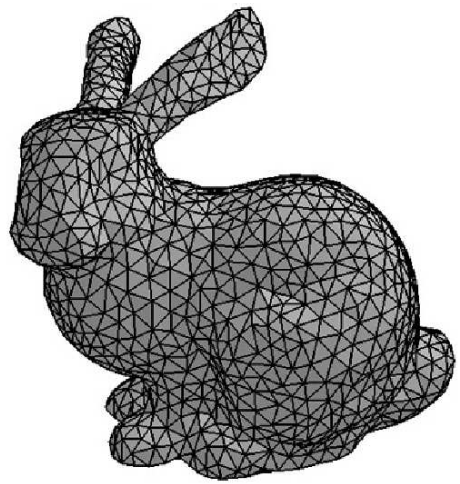

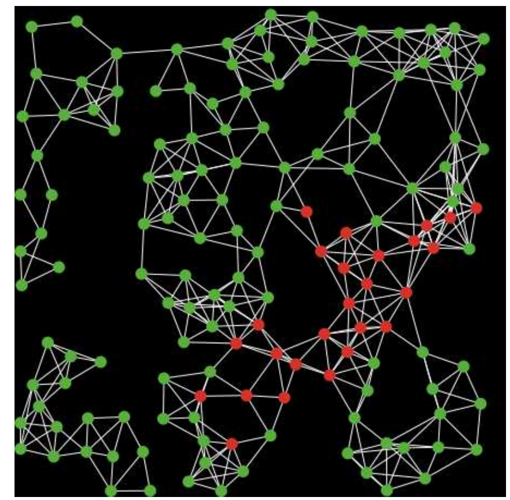

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Attention: Data so far only in Euclidean spaces!

The graph framework

Idea: Notion of a derivative for vertex functions $[2, 3]$.

$$\nabla f(u,v) = \sqrt{w(u,v)}\,\big(f(v) - f(u)\big)$$

$[2]$ A. Elmoataz, O. Lézoray, S. Bougleux: *Nonlocal Discrete Regularization on Weighted Graphs: A Framework for Image and Manifold Processing*. IEEE TIP 17 (2008)
$[3]$ G. Gilboa, S. Osher: *Nonlocal operators with applications to image processing*. Multiscale Modeleling and Simulation 7 (2008)

The graph framework

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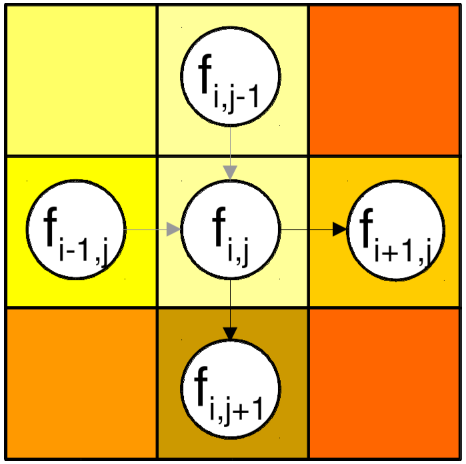
$$\nabla f(u,v) = \sqrt{w(u,v)}\,(f(v) - f(u))$$

Special case: Finite forward differences

Let $G = (V, E, w)$ be a directed 2-neighbour grid graph with the weight function w chosen as:

$$w(u,v) = \begin{cases} \frac{1}{h^2}\ , & \text{if } u \sim v \\ 0\ , & \text{else} \end{cases}$$



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Translating variational problems to graphs

Idea:

Solve variational problems on graphs using convex optimization.

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Example: Rudin-Osher-Fatemi total variation (TV) denoising model [4]

Find a minimizer $f : V \to \mathbb{R}$ of the energy functional

$$E(f) \;=\; \lambda||f - f_0||^2 + ||f||_{TV} \;, \quad \lambda > 0$$

[4] L.I. Rudin, S. Osher, E. Fatemi: *Nonlinear total variation based noise removal algorithms*. Physica D 60: 259–268 (1992)

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$$E(f) \; = \; \lambda ||f - f_0||^2 + ||f||_{TV} \;, \quad \lambda > 0$$

with $$||f||_{TV} \; = \; \sum_{u \in V} \left(\sum_{v \sim u} ||\nabla f(u,v)||^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

[4] L.I. Rudin, S. Osher, E. Fatemi: *Nonlinear total variation based noise removal algorithms.* Physica D 60: 259–268 (1992)

Translating higher order differential operators

Idea:

Mimic important PDEs from image processing on finite weighted graphs.

Translating higher order differential operators

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Mimic important PDEs from image processing on finite weighted graphs.

Example: The p-Laplace equation

Let $\Omega \subset \mathbb{R}^n$ an open, bounded set, let $1 \leq p < \infty$ and $f \colon \Omega \to \mathbb{R}$. We are interested in a solution of the homogeneous p-Laplace equation:

$$\begin{aligned}\Delta_p f(x) &= -\operatorname{div}\left(\left|\frac{\partial f}{\partial x_i}\right|^{p-2} \frac{\partial f}{\partial x_i}\right)(x) \\ &= -\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\left|\frac{\partial f}{\partial x_i}\right|^{p-2} \frac{\partial f}{\partial x_i}\right)(x) = 0\end{aligned}$$

Translating higher order differential operators

Idea:
Mimic important PDEs from image processing on finite weighted graphs.

Example: The graph p-Laplace equation [5]

Let $G(V, E, w)$ a finite weighted graph, let $1 \leq p < \infty$ and $f : V \to \mathbb{R}$ a vertex function.
We are interested in a solution of the following finite difference equation:

$$\begin{aligned} \Delta_p f(u) &= \frac{1}{2} \operatorname{div} \left(||\nabla f||^{p-2} \nabla f \right)(u) \\ &= -\sum_{v \sim u} (w(u,v))^{p/2} |f(v) - f(u)|^{p-2} (f(v) - f(u)) \;=\; 0 \end{aligned}$$

[5] A. Elmoataz, M. Toutain, D. Tenbrinck: *On the p-Laplacian and ∞-Laplacian on Graphs with Applications in Image and Data Processing.* SIAM Journal on Imaging Sciences 8 (2016)

The graph framework

Observation:

This framework enables the translation of local/nonlocal PDEs and variational models to any graph-structured data.

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Original 3D point cloud

Noisy 3D point cloud

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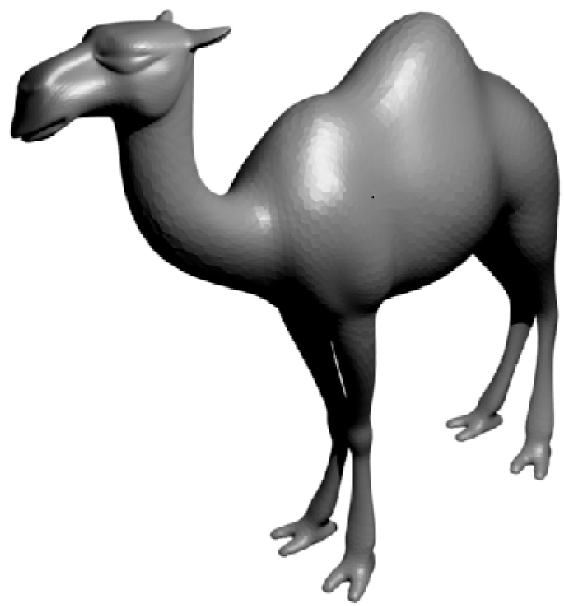
Original 3D point cloud



Denoised 3D point cloud

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3D point cloud of a scanned person

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3D point cloud of a scanned person

User-defined region for color inpainting

The graph framework

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3D point cloud of a scanned person

Result of color inpainting (local)

The graph framework

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3D point cloud of a scanned person

Result of color inpainting (nonlocal)

Related work

Nonlocal methods:

- A. Buades: *A Nonlocal Algorithm for Image Denoising*. CVPR (2005)
- G. Gilboa, S. Osher: Nonlocal Operators with Applications to Image Processing, Multiscale Model. Simul. 7(3) (2008)
- P. Arias, V. Caselles, G. Sapiro: *A variational framework for non-local image inpainting*. Energy Minimization Methods in Computer Vision and Pattern Recognition (2009)
- X. Zhang, M. Burger, X. Bresson, S. Osher: Bregmanized nonlocal regularization for deconvolution and sparse reconstruction, SIAM Journal on Imaging Sciences 3, (2010)
- J.F. Aujol, G. Gilboa, N. Papadakis: Nonlocal Total Variation Spectral Framework. Proc. Scale Space and Variational Methods in Computer Vision - SSMV (2015)
- A. Chambolle, M. Morini, M. Ponsiglione: Nonlocal Curvature Flows. Arch Rational Mech Anal 218 (2015)
- J. Lellmann, K. Papafitsoros, C.-B. Schönlieb, D. Spector: *Analysis and Application of a Nonlocal Hessian*. SIAM Journal on Imaging Sciences 8, (2015)
- Z. Shi, S. Osher, W. Zhu: *Weighted Nonlocal Laplacian on Interpolation from Sparse Data*. J. Sci. Comp. 3 (2017)

Related work

Finite weighted graphs:

- A. Elmoataz, O. Lézoray, S. Bougleux: *Nonlocal Discrete Regularization on Weighted Graphs: a Framework for Image and Manifold Processing*. IEEE TIP 17 (2008)
- F. Lozes, A. Elmoataz, O. Lezoray: *Partial Difference Operators on Weighted Graphs for Image Processing on Surfaces and Point Clouds*. IEEE TIP 23 (2014)
- Y. van Gennip, N. Guillen, B. Osting, A.L. Bertozzi: Mean Curvature, Threshold Dynamics, and Phase Field Theory on Finite Graphs. Milan Journal of Mathematics 82 (2014)
- E. Merkurjev, E. Bae, A.L. Bertozzi, X.C. Tai: *Global Binary Data Optimization on Graphs for Data Segmentation*. J. Math. Imag. Vis. 52 (2015)
- L. Landrieu, G. Obozinski: *Cut Pursuit: Fast Algorithms to Learn Piecewise Constant Functions on General Weighted Graphs*. SIAM SIIMS 10 (2017)

Transition from discrete to continuous mathematics:

- M. Belkin, P. Niyogi: *Towards a Theoretical Foundation for Laplacian-Based Manifold Methods*. J. Comput. System Sci. 74 (2008)
- U. von Luxburg, M. Belkin, O. Bousquet: *Consistency of spectral clustering*. The Annals of Statistics 36 (2008)
- N. Garcia Trillos, D. Slepcev: *Continuum Limit of Total Variation on Point Clouds*. Arch. Rat. Mech. An. 1 (2016)
- M. Thorpe, F. Theil: *Asymptotic Analysis of the Ginzburg-Landau Functional on Point Clouds*. arXiv:1604.04930 (2017)
- J. Calder, N. García Trillos: *Improved spectral convergence rates for graph Laplacians on ϵ-graphs and k-NN graphs*. arXiv: 1910.13476 (2019)
- T. Roith: Continuum Limit of Lipschitz Learning on Graphs, Master thesis at FAU. (2020)

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The basic idea

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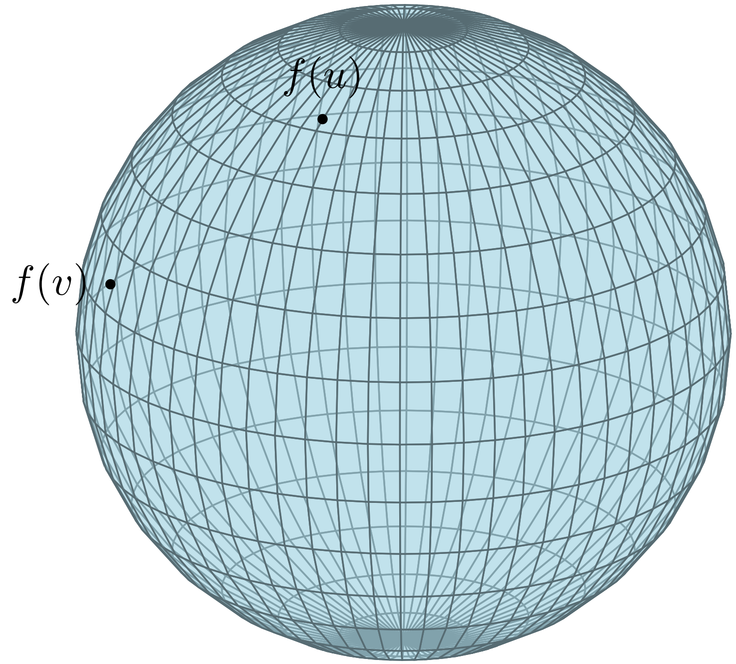

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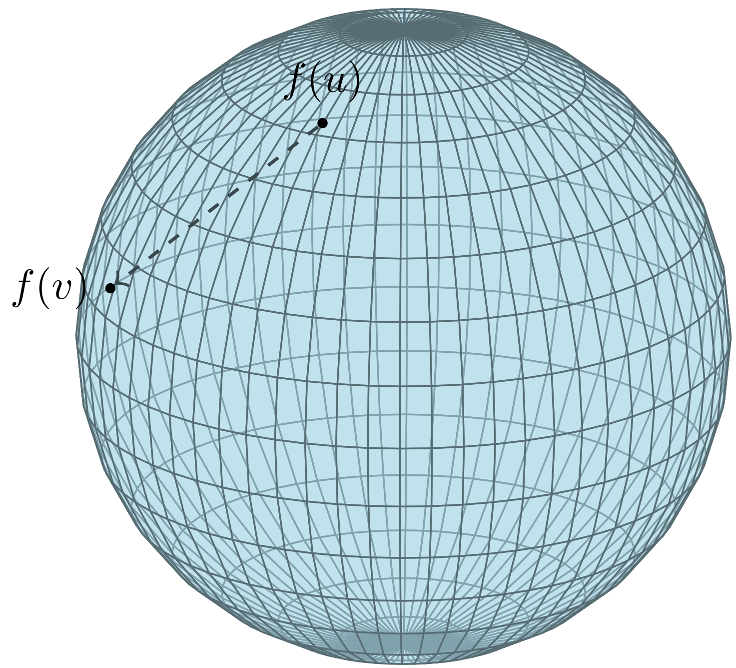

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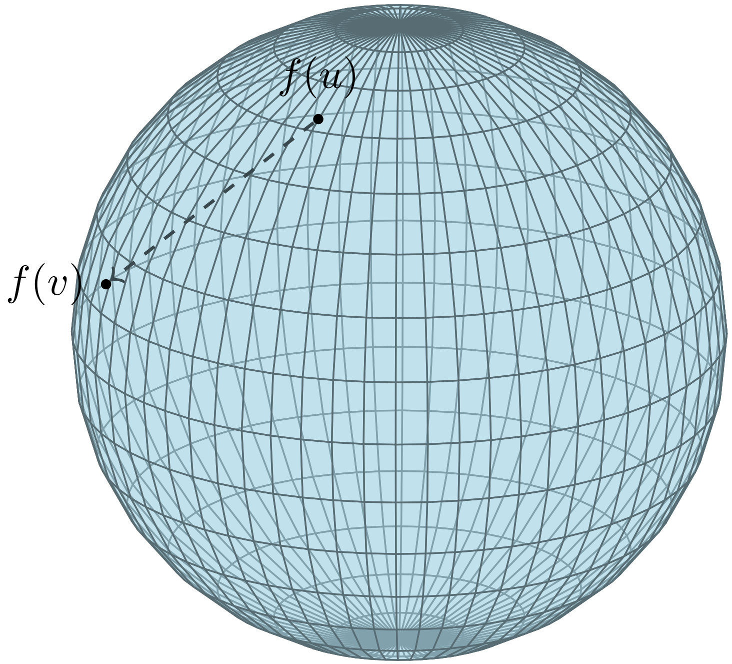

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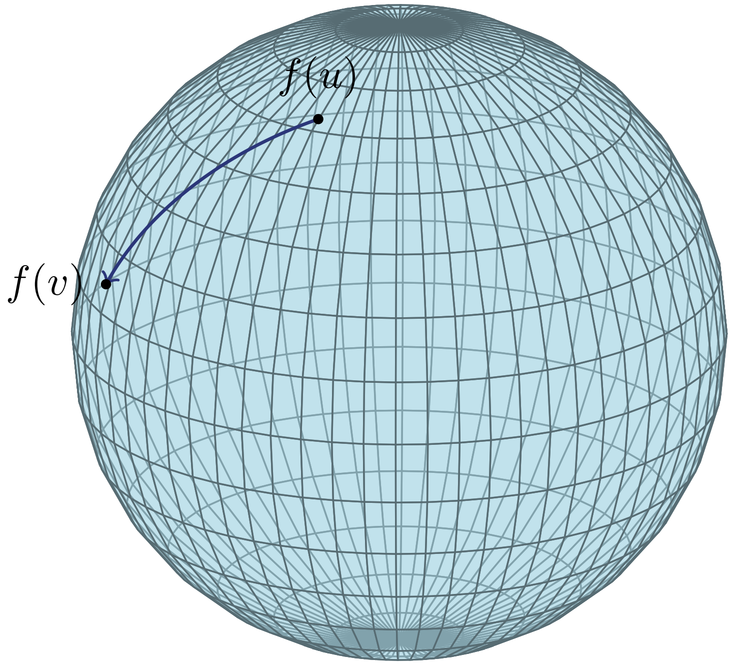

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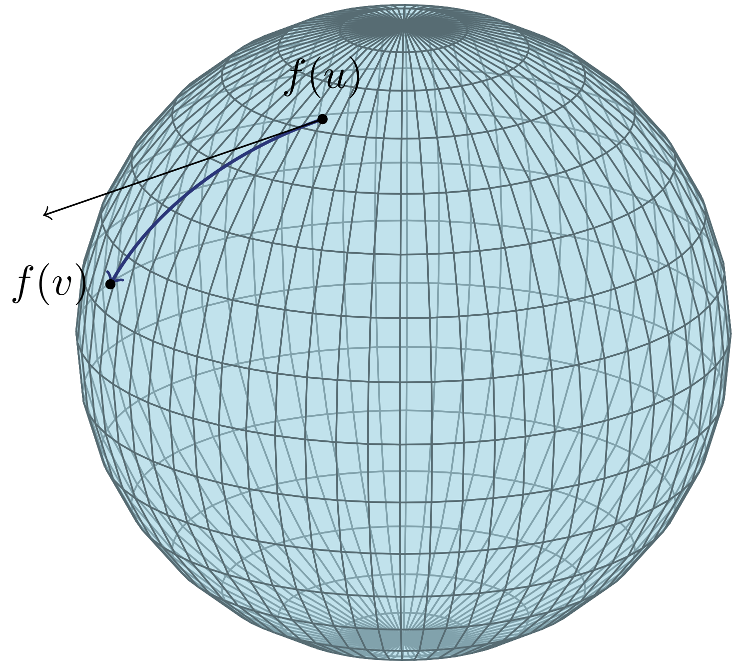

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Idea: We compute differences in the respective tangential space.

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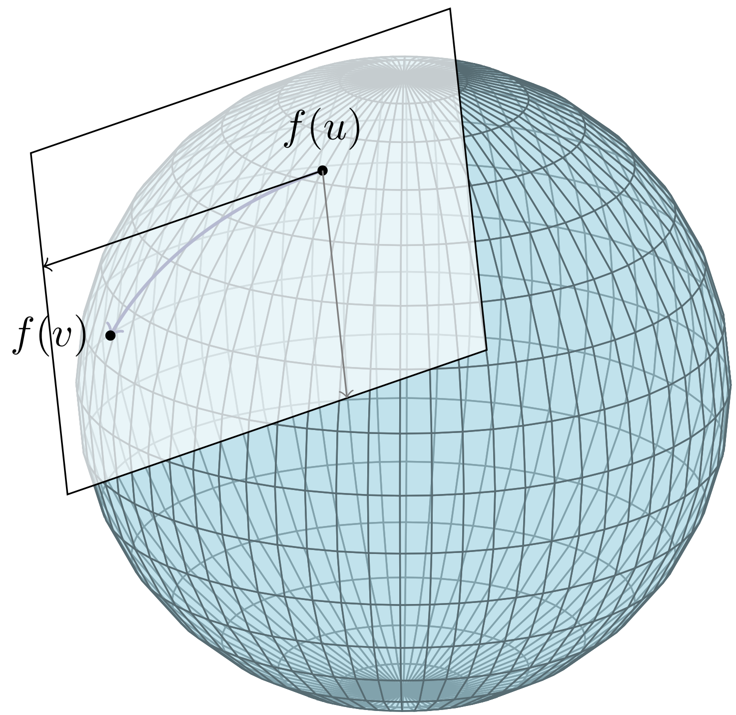

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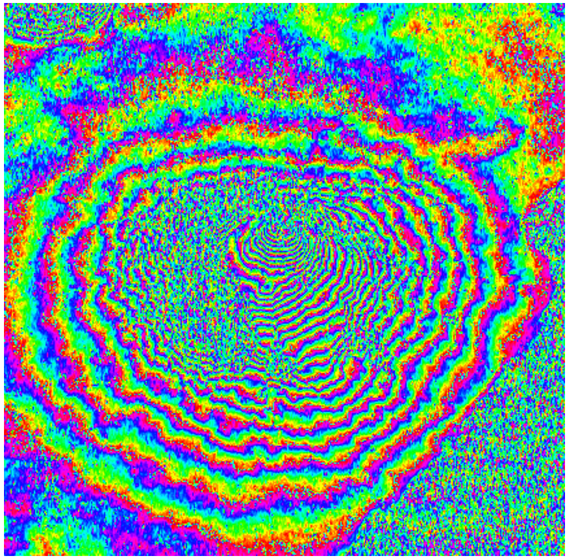
- Interferometric synthetic aperture radar (InSAR) imaging



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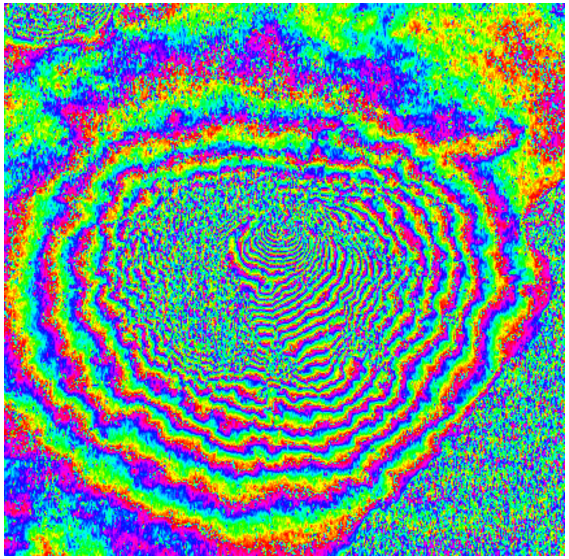
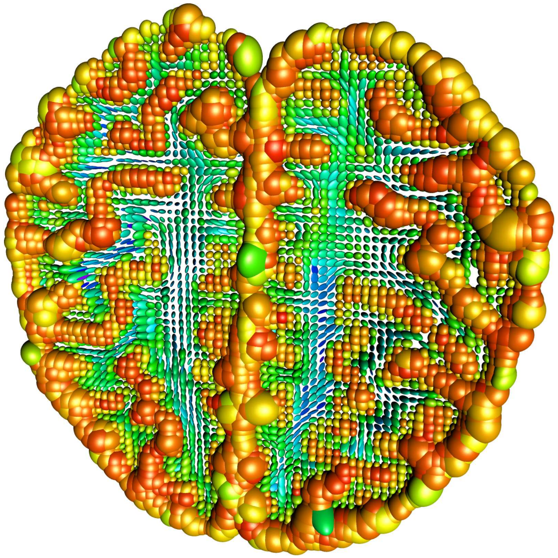
Why should one work with manifold-valued data?

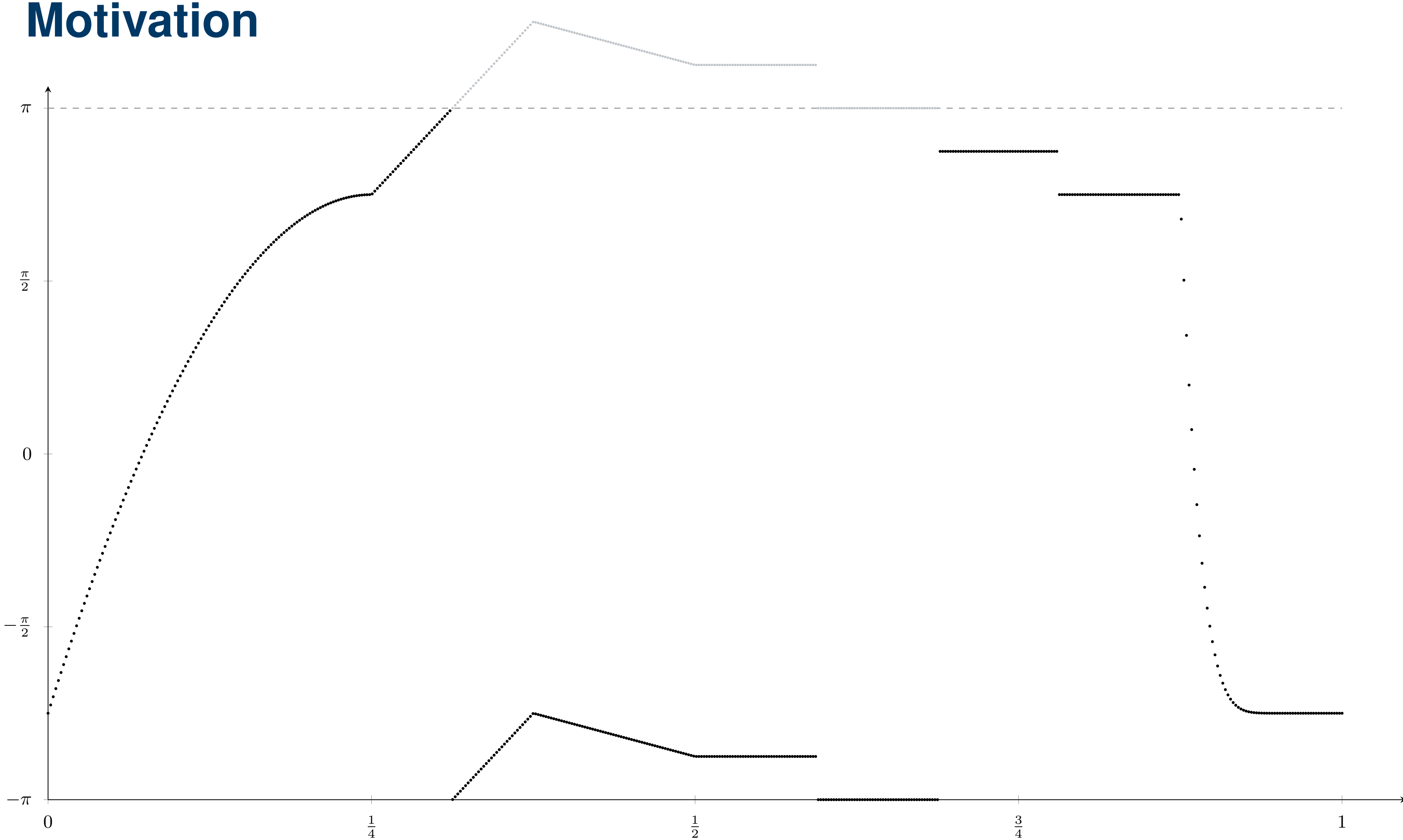
$\rightarrow$ There are many interesting applications with values on manifolds, e.g.,

- Interferometric synthetic aperture radar (InSAR) imaging
- Diffusion tensor imaging (DT-MRI)





Motivation



- **Observation:** Phase signal $f: [0,1] \to [-\pi, \pi]$ is wrapped

Department MATHEMATIK

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

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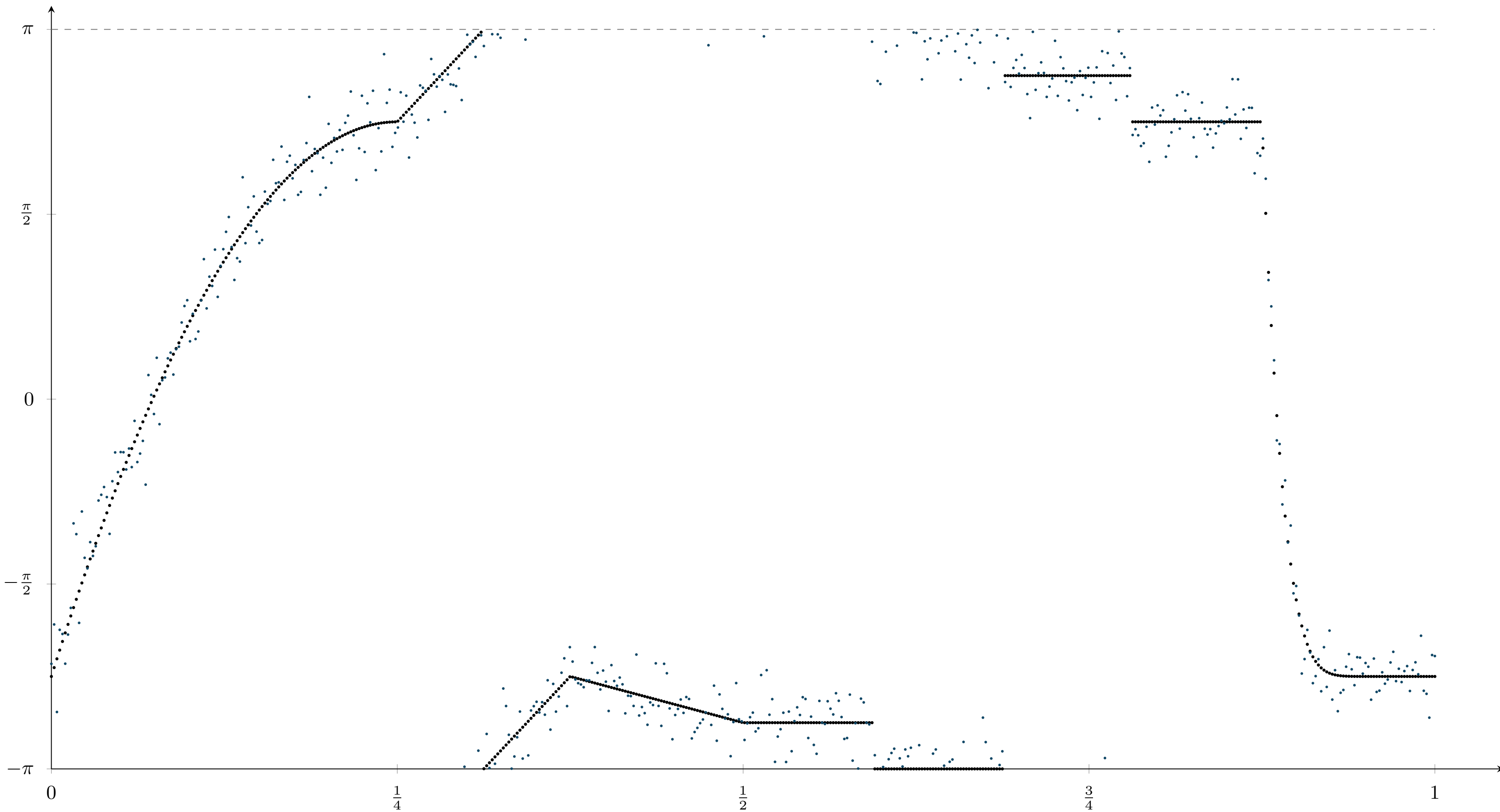

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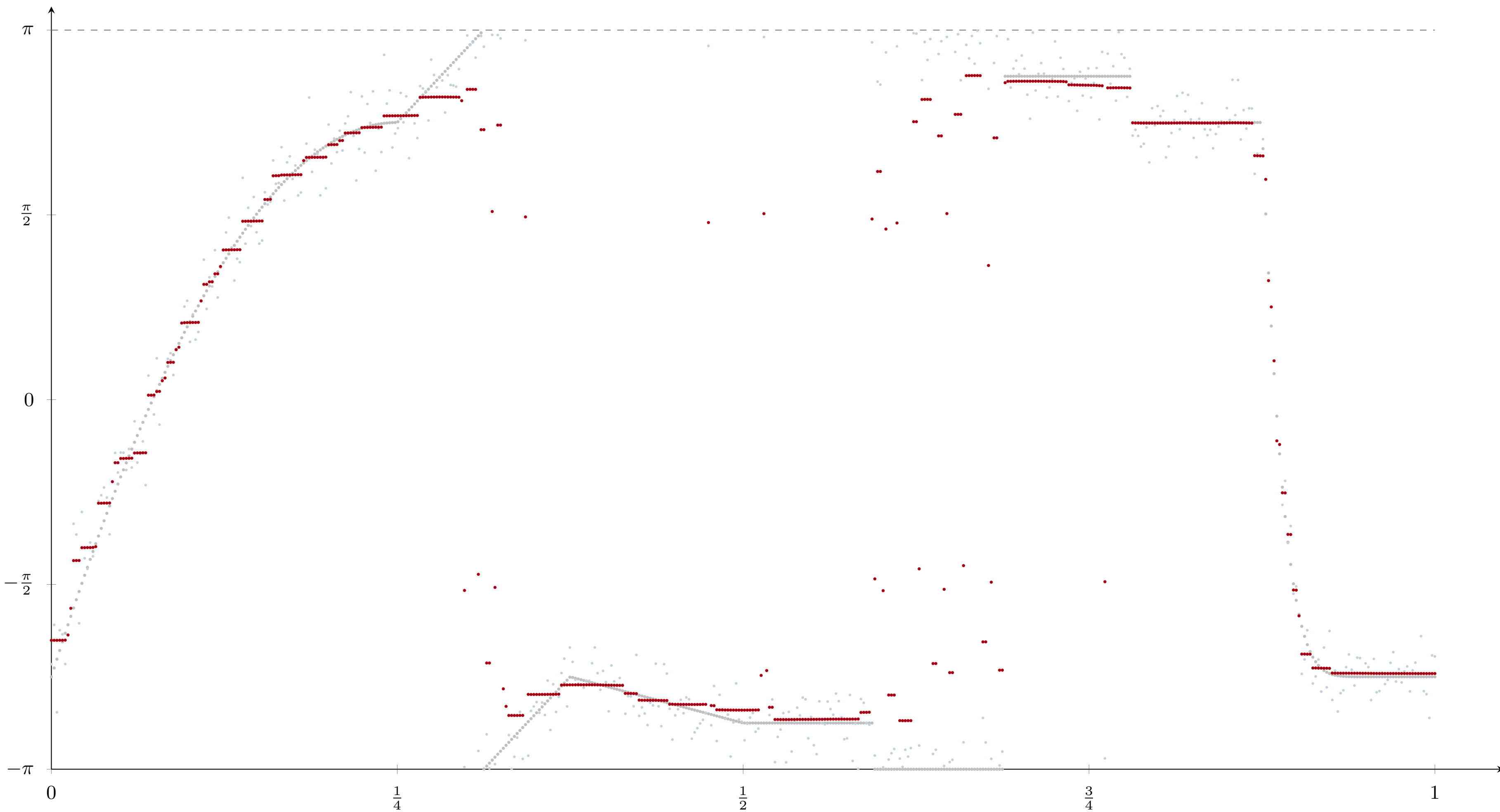
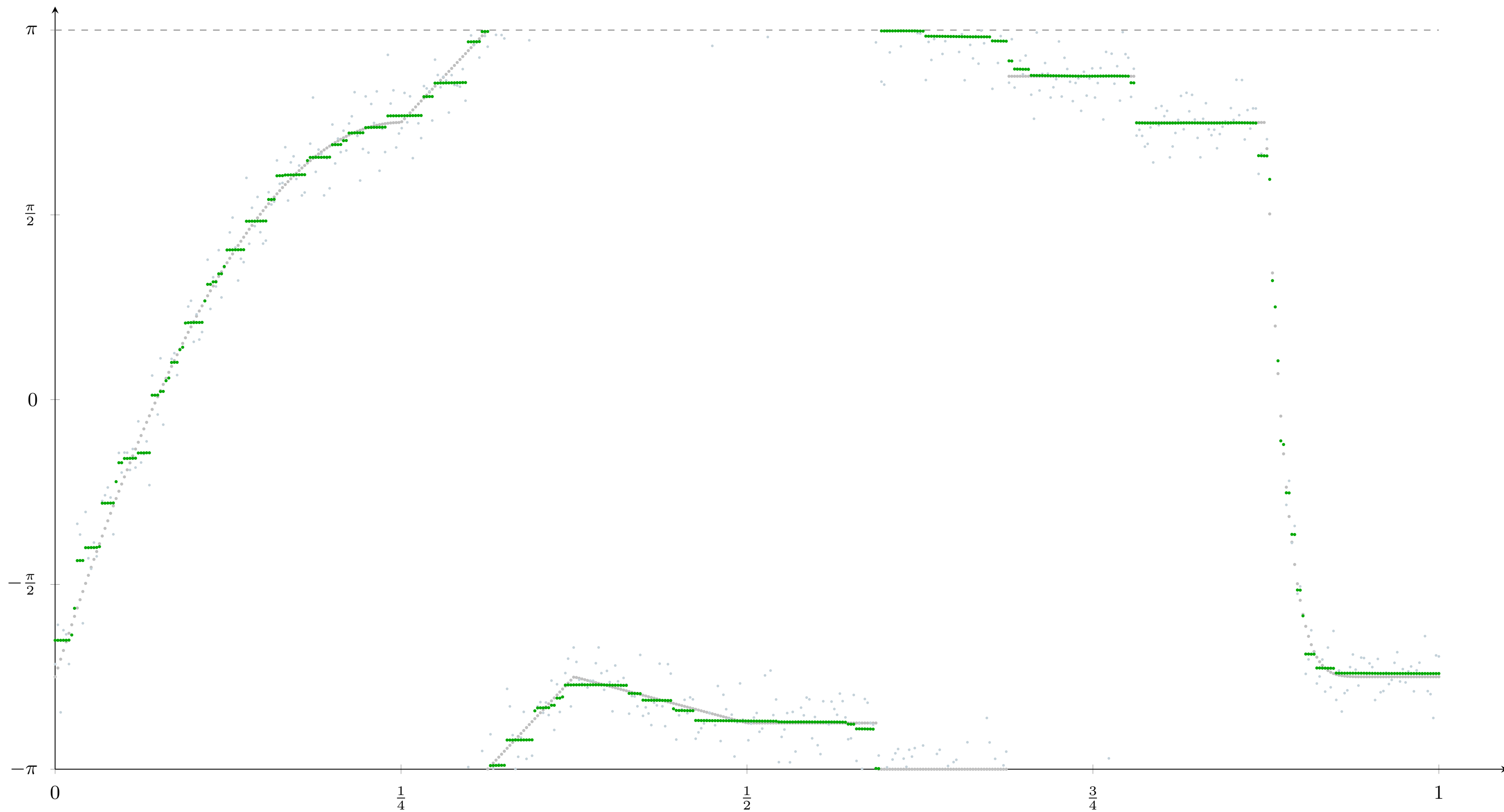


- **Problem:** Traditional TV denoising is not feasible ($\rightarrow$ large jumps)

Motivation



- **Solution:** Perform TV denoising on the Riemannian manifold $\mathbb{S}^1$

Related work on manifold-valued images

Books on optimization on manifolds:

- N. Bournal: *An Introduction to Optimization on Smooth Manifolds*, Available online, 2020.
- M. Bačák: *Convex analysis and optimization in Hadamard spaces*, De Gruyter, 2014.
- P.-A. Absil, R. Mahony, R. Sepulchre: *Optimization Algorithms on Matrix Manifolds*, Princeton University Press, 2008.
- C. Udrişte: *Convex Functions and Optimization Methods on Riemannian Manifolds*, Springer, 1994.

Total Variation & Image Processing on Manifolds:

- K. Bredies, M. Holler, M. Storath, A. Weinmann: *Total Generalized Variation for Manifold-valued Data*, arXiv:1709.01616, 2017.
- P.-A. Absil, P.-Y. Gousenbourger, P. Striewski, B. Wirth: *Differentiable Piecewise-Bézier Surfaces on Riemannian Manifolds*. SIAM J. Imaging Sci. (2016)
- M. Bačák, R. Bergmann, G. Steidl, A. Weinmann: *Second order non-smooth variational model for restoring manifold-valued images*, SIAM J. Sci. Comput., 2016.
- R. Bergmann, R. H. Chan, R. Hielscher, J. Persch and G. Steidl: *Restoration of manifold-valued images by half-quadratic minimization*, Inv. Probl. and Imag., 2016
- A. Weinmann, L. Demaret, M. Storath: *Total variation regularization for manifold-valued data*, SIAM J. Imag. Sci., 2014.
- J. Lellmann, E. Strekalovskiy, S. Koetter, D. Cremers: *Total variation regularization for functions with values in a manifold*, IEEE ICIV, 2013.
- X. Pennec, P. Fillard, and N. Ayache. *A Riemannian framework for tensor computing*, International Journal of Computer Vision , 2006

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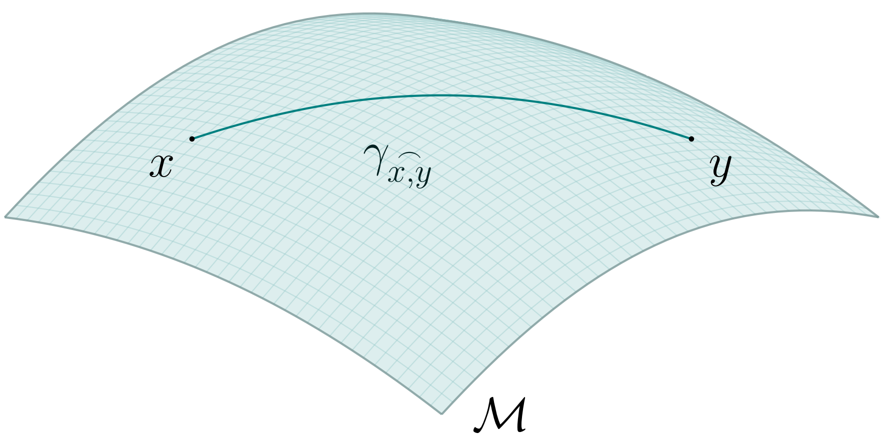
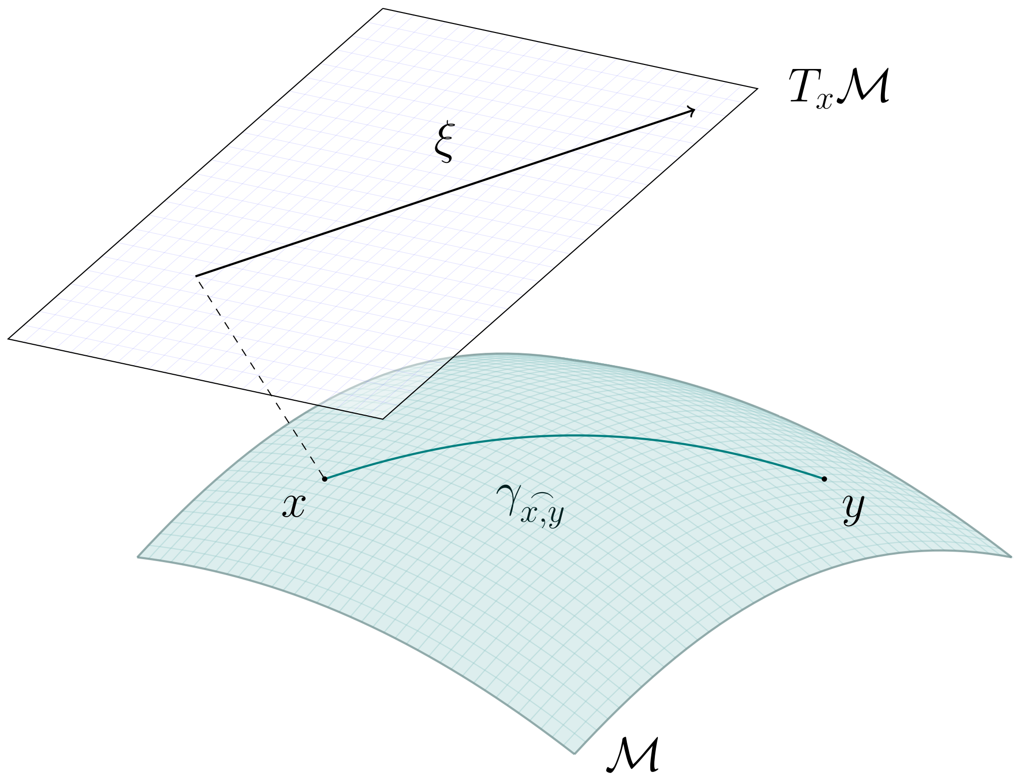
Notations on a Riemannian manifold



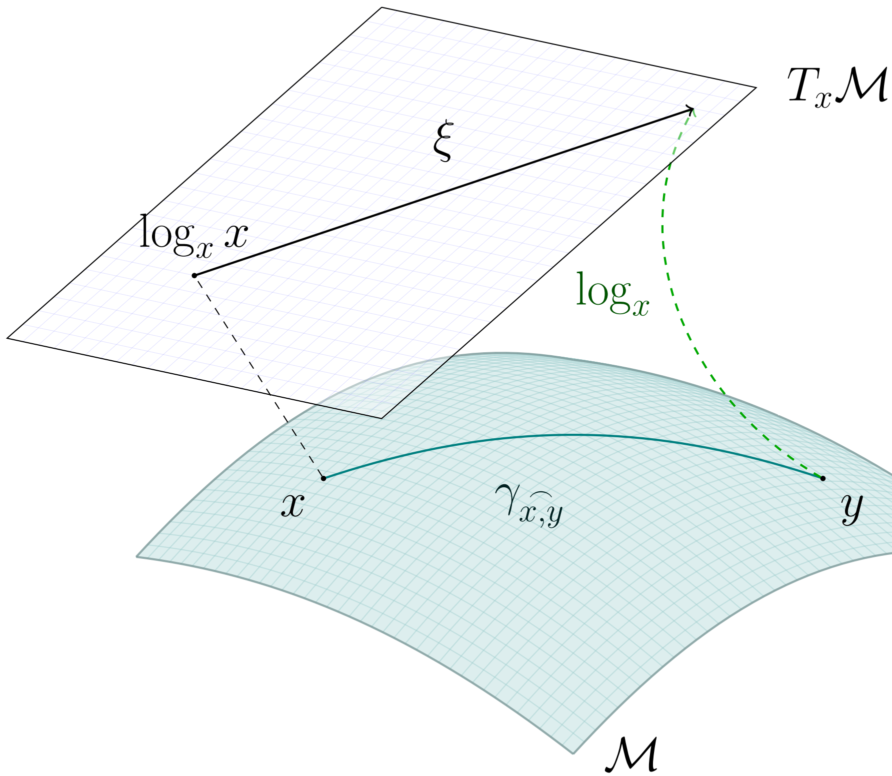
- m-dimensional complete Riemannian manifold $\mathcal{M}$
- geodesic $\gamma_{\widehat{x,y}}$ on $\mathcal{M}$ connecting $x, y \in \mathcal{M}$

Notations on a Riemannian manifold



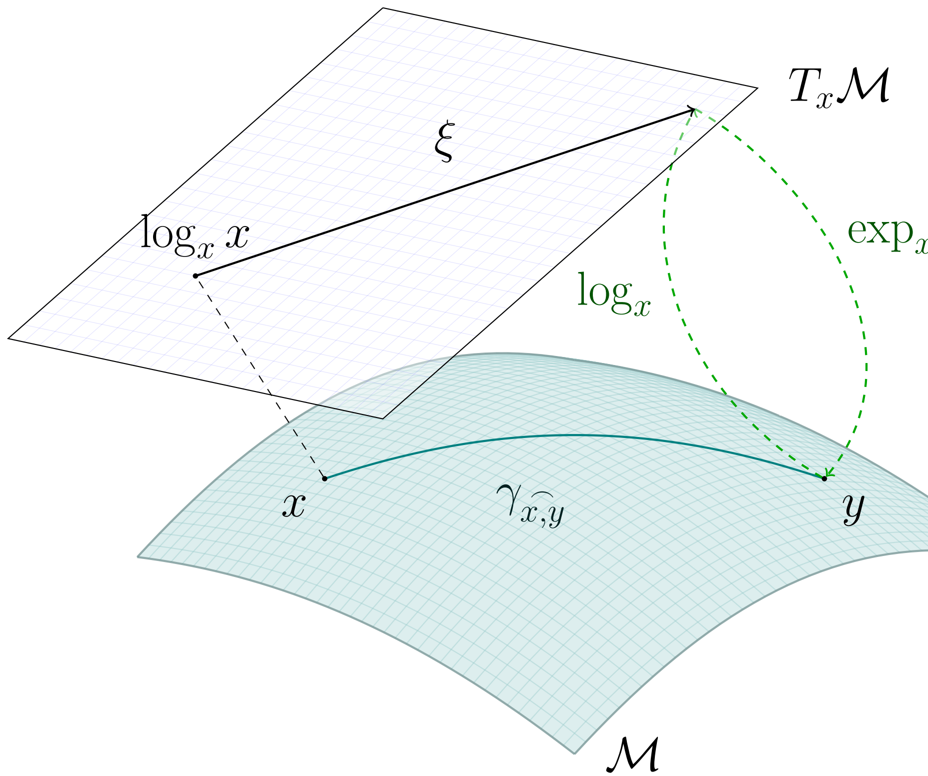
- tangential plane $\mathrm{T}_x\mathcal{M}$ at base point $x \in \mathcal{M}$

Notations on a Riemannian manifold



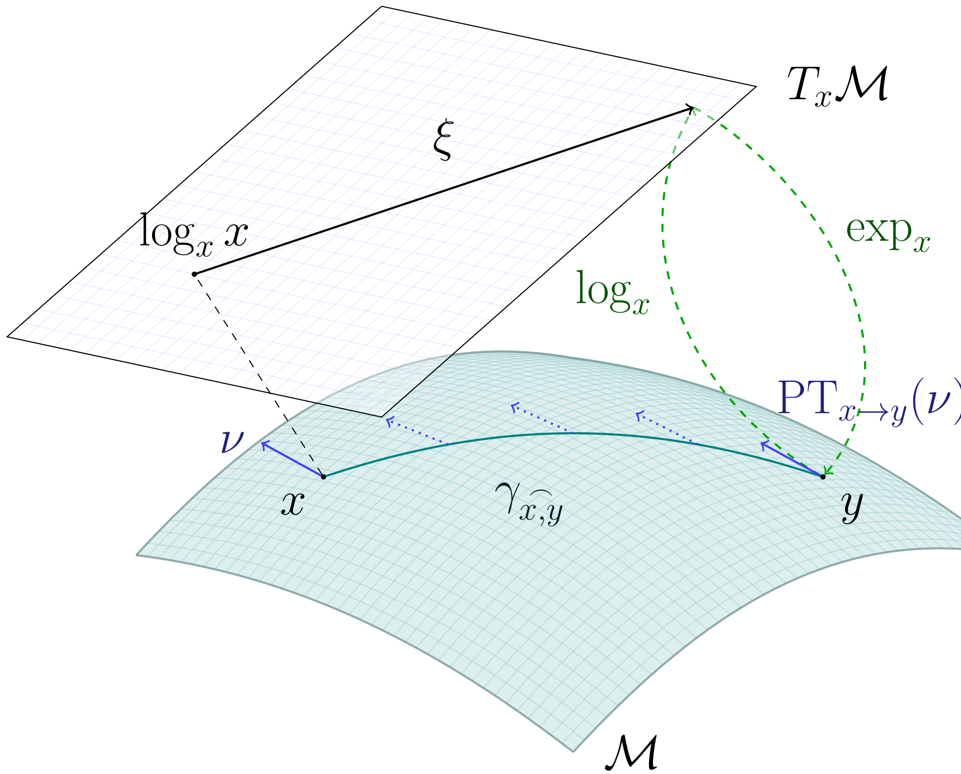
- logarithmic map $\log_x y \colon \mathcal{M} \to T_x\mathcal{M}$ with $\log_x y = \dot{\gamma}_{\widehat{x,y}}(0)$

Notations on a Riemannian manifold



- exponential map $\exp_x \xi = \gamma(1) = y$ with $\gamma(0) = x$ and $\dot\gamma(0) = \xi$

Notations on a Riemannian manifold



- parallel transport $\mathrm{PT}_{x\to y}(\nu)$ of a tangential vector $\nu \in \mathrm{T}_x\mathcal{M}$ along $\gamma_{\widehat{x,y}}$

Function spaces

Observation:

Let $f, g \colon V \to \mathcal{M}$ and $\lambda \in \mathbb{R}$. There is no reasonable definition for mathematical expressions like $f + g$ or $\lambda \cdot f$.

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Definition: Space of vertex functions

The set of manifold-valued vertex functions

$$\mathcal{H}(V;\mathcal{M}) \coloneqq \{f\colon V \to \mathcal{M}\}$$

induces a **metric space** with the metric:

$$\begin{aligned} d_{\mathcal{H}(V;\mathcal{M})}(f,g) &\coloneqq \sum_{u \in V} \langle \log_{f(u)} g(u), \log_{f(u)} g(u) \rangle_{f(u)} \\ &= \sum_{u \in V} d_{\mathcal{M}}(f(u), g(u)) \end{aligned}$$

Function spaces

Definition: Space of edge functions

Let $f \in \mathcal{H}(V;\mathcal{M})$ be a manifold-valued vertex function. Then the set of affiliated edge functions

$$\mathcal{H}(E;\mathrm{T}_f\mathcal{M}) \;\coloneqq\; \{H_f \colon E \to \mathrm{T}\mathcal{M} \text{ with } H_f(u,v) \in \mathrm{T}_{f(u)}\mathcal{M}\}$$

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Observation:

Edge functions are in general not symmetric wrt. $u, v \in V$, i.e.,

$$\mathrm{T}_{f(u)}\mathcal{M} \ni H_f(u,v) \;\neq\; H_f(v,u) \in \mathrm{T}_{f(v)}\mathcal{M}.$$

Weighted local gradient

Idea:

Use local tangential spaces to measure distances between manifold values.

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Definition: Weighted local gradient operator

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$$\nabla f(u,v) \;:=\; \sqrt{w(u,v)}\log_{f(u)} f(v).$$

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Observation:

The weighted local gradient is antisymmetric wrt. parallel transport, i.e.,

$$\begin{aligned}\nabla f(u,v) &= \sqrt{w(u,v)}\,\log_{f(u)} f(v)\\ &= -\sqrt{w(u,v)}\,\mathrm{PT}_{f(v)\to f(u)} \log_{f(v)} f(u)\end{aligned}$$

Towards a divergence operator

Question:

Given the local weighted gradient operator $\nabla\colon \mathcal{H}(V;\mathcal{M}) \to \mathcal{H}(E;\mathrm{T}\mathcal{M})$, can we give a meaningful definition of a divergence?

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We would like to find a global adjoint operator $\nabla^*\colon \mathcal{H}(E;\mathrm{T}\mathcal{M}) \to \mathcal{H}(V;\mathcal{M})$, which is specified by the relation:

$$\langle \nabla f, H_f \rangle \;=\; \langle f, \nabla^* H_f \rangle \quad \text{for all} \;\; H_f \in \mathcal{H}(E;\mathrm{T}_f\mathcal{M})$$

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Observation:

We are not able to derive a meaningful definition of the inner product on the right side of this equation. ☹

Local weighted divergence

Proposition

For $f \in \mathcal{H}(V;\mathcal{M})$ and $H_f \in \mathcal{H}(E;\mathrm{T}_f\mathcal{M})$ we have

$$\langle \nabla_w f, H_f \rangle_{\mathcal{H}(E;\mathrm{T}_f\mathcal{M})} = \sum_{u\in V}\sum_{v\sim u} \langle \log_{f(u)} f(v), -\operatorname{div} H_f(u) \rangle_{f(u)},$$

with the local weighted divergence $\operatorname{div}\colon \mathcal{H}(E;\mathrm{T}\mathcal{M}) \to \mathcal{H}(V;\mathrm{T}\mathcal{M})$ given as

$$\operatorname{div} H_f(u) := \frac{1}{2}\sum_{v\sim u} \sqrt{w(v,u)}H_f(v,u) - \sqrt{w(u,v)}H_f(u,v)$$

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Corollary

If $H_f \in \mathcal{H}(E;\mathrm{T}_f\mathcal{M})$ is antisymmetric wrt. the parallel transport and $w\colon E \to [0,1]$ is a symmetric weight function, we get

$$\operatorname{div} H_f(u) = -\sum_{v \sim u} \sqrt{w(u,v)} H_f(u,v).$$

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Towards a Laplace operator

Observation:

Let us assume a symmetric weighting function, i.e., $w(u,v) = w(v,u)$. Then, if we put

$$\operatorname{div}(\nabla f)(u) \; = \; \frac{1}{2} \sum_{v \sim u} \sqrt{w(v,u)} \underbrace{\mathrm{PT}_{f(v) \to f(u)} \nabla f(v,u))}_{= -\nabla f(u,v)} - \sqrt{w(u,v)} \nabla f(u,v)$$

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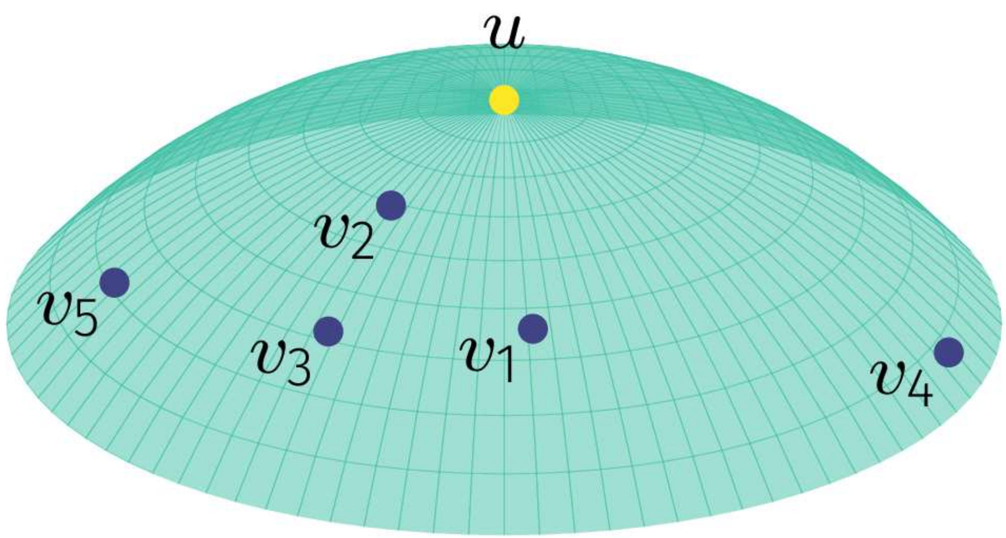
we derive a discrete graph Laplace operator as the local weighted mean of the neighbor values projected into the tangential plane $\mathcal{T}_{f(u)}\mathcal{M}$.

Illustration of the manifold-valued Laplace operator



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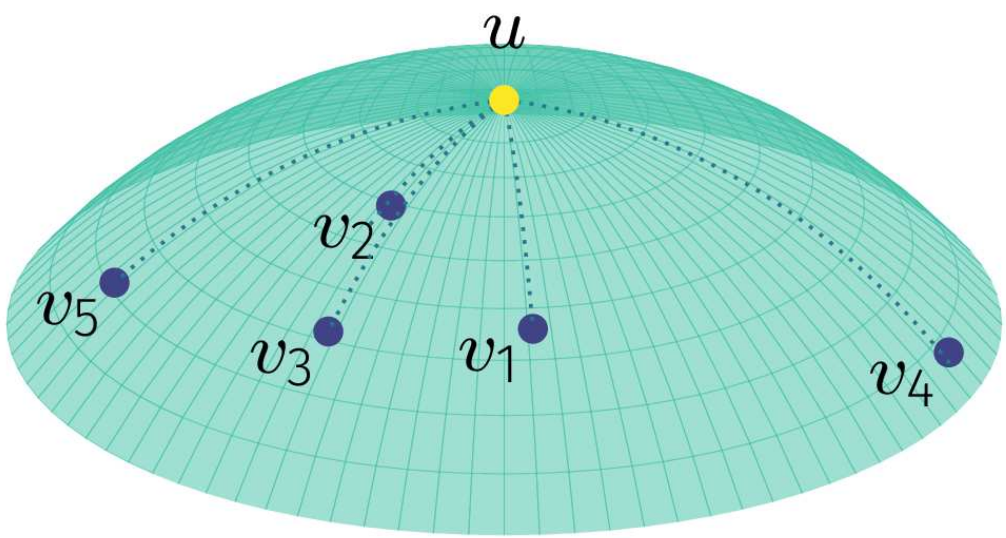

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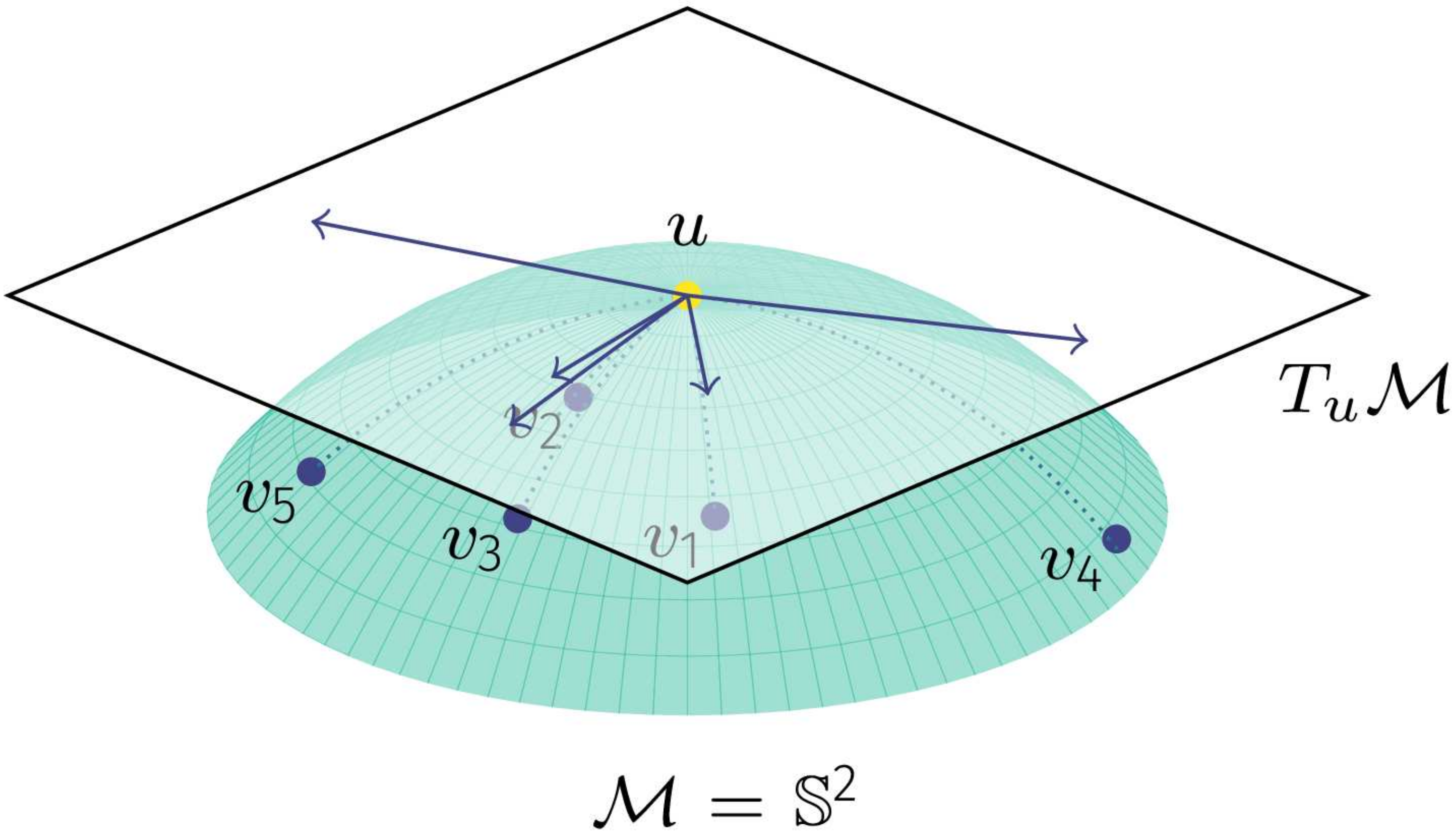

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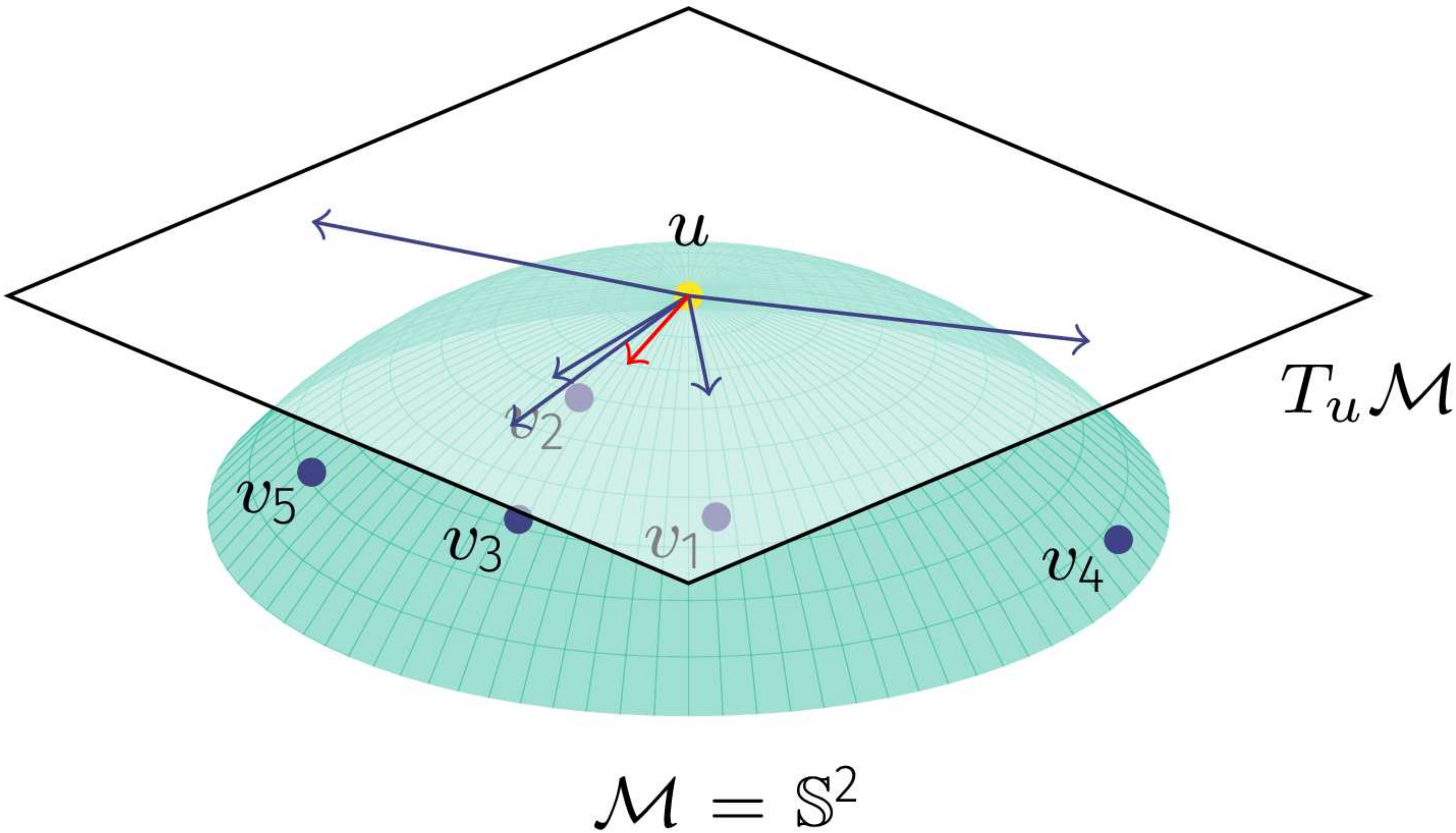

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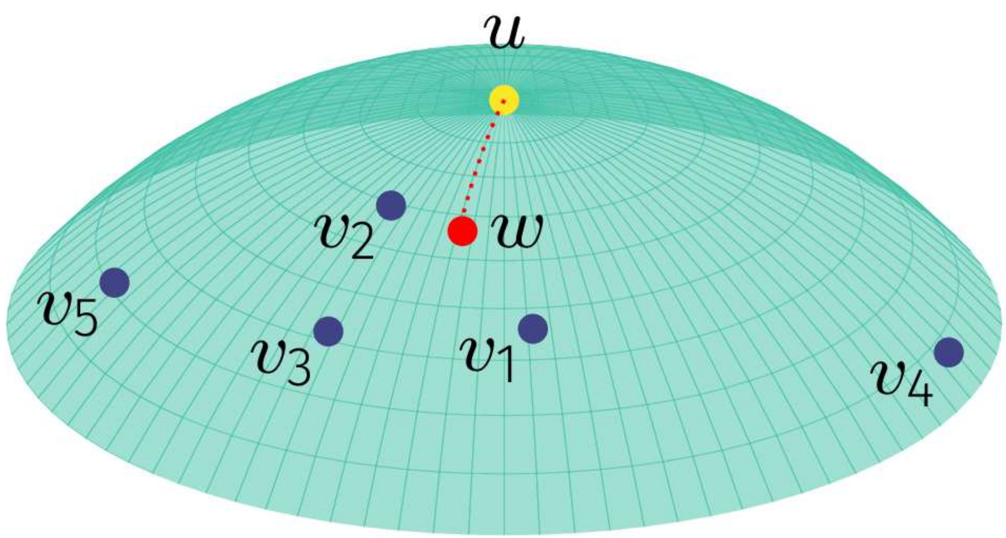

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$$\begin{aligned} \Delta_p^{\mathrm{a}} f(u) &:= \operatorname{div}\big(\|\nabla f\|_{f,p}^{p-2} \nabla f\big)(u) \\ &= -\sum_{v \sim u} \sqrt{w(u,v)}^{\,p} d_{\mathcal{M}}^{p-2}(f(u), f(v)) \log_{f(u)} f(v) \end{aligned}$$

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Remember: These are elements in $T_{f(u)}\mathcal{M}$.

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Given: Manifold-valued (noisy) data $f_0 \colon V \to \mathcal{M}$

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Aim: Denoise f_0 by solving the following optimization problem:

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Note that for $\lambda > 0$ this formulation covers two well-known special cases:

p=1: (Anisotropic) total variation-regularized denoising $[6, 7]$
p=2: Tykhonov-regularized denoising

$[6]$ J. Lellmann, E. Strekalovskiy, S. Koetter, D. Cremers: *Total variation regularization for functions with values in a manifold*. ICCV (2013)
$[7]$ A. Weinmann, L. Demaret, M. Storath: *Total variation regularization for manifold-valued data*. SIAM Journal on Imaging Sciences 7 (2014)

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We need to find minimizers of the (anisotropic) energy functional:

$$\mathcal{E}_{\mathrm{a}}(f) \;\coloneqq\; \frac{\lambda}{2} \sum_{u \in V} d^2_{\mathcal{M}}(f_0(u), f(u)) \;+\; \frac{1}{p} \sum_{(u,v) \in E} \|\nabla_w f(u,v)\|^p_{f(u),p}$$

Variational denoising model

Based on the notion of the subdifferential on $\mathcal{M}$ we derive:

$$
\begin{aligned}
0 \in \partial \left(\frac{\lambda}{2} \sum_{u \in V} d_{\mathcal{M}}^2(f_0(u), f(u)) + \frac{1}{p} \sum_{(u,v) \in E} \|\nabla f\|_{f(u)}^p \right)(f) \\
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= -\lambda \sum_{u \in V} \log_{f(u)} f_0(u) - \sum_{u \in V} \sum_{v \sim u} (w(u,v))^{\frac{p}{2}} \, d_{\mathcal{M}}^{p-2}(f(u), f(v)) \log_{f(u)} f(v) \\
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\end{aligned}
$$

Deriving the above necessary optimality conditions one has to solve the following PDE under suitable boundary conditions:

$$\Delta^{\mathrm{a}}_p f(u) - \lambda\log_{f(u)} f_0(u) = 0 \in T_{f(u)}\mathcal{M} \quad \text{for all } u \in V$$

Summary of the basic idea

Vector-valued case $f : V \to \mathbb{R}^n$

Manifold-valued case $f : V \to \mathcal{M}$

Summary of the basic idea

Vector-valued case $f : V \to \mathbb{R}^n$

Space of vertex functions:

$\mathcal{H}(V; \mathbb{R}^n)$ is a Euclidean space

Manifold-valued case $f : V \to \mathcal{M}$

$\mathcal{H}(V; \mathcal{M})$ is a metric space

Summary of the basic idea

Vector-valued case $f: V \to \mathbb{R}^n$

Space of vertex functions:

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Gradient operator:

$$\nabla f(u,v) := \sqrt{w(u,v)}\big(f(v) - f(u)\big)$$

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Summary of the basic idea

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Graph p-Laplacian operator:

$$\Delta_p^{\mathrm{a}} f(u) = -\sum_{v\sim u} \sqrt{w(u,v)}^p \|f(v) - f(u)\|^{p-2}(f(v) - f(u))$$

Manifold-valued case $f : V \to \mathcal{M}$

$\mathcal{H}(V;\mathcal{M})$ is a metric space

$$\nabla f(u,v) := \sqrt{w(u,v)}\log_{f(u)} f(v)$$

$$\Delta_p^{\mathrm{a}} f(u) = -\sum_{v\sim u} \sqrt{w(u,v)}^p d_{\mathcal{M}}^{p-2}(f(u), f(v)) \log_{f(u)} f(v)$$

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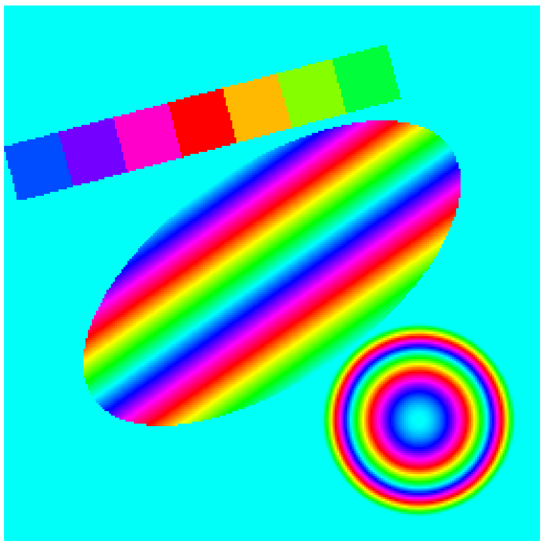
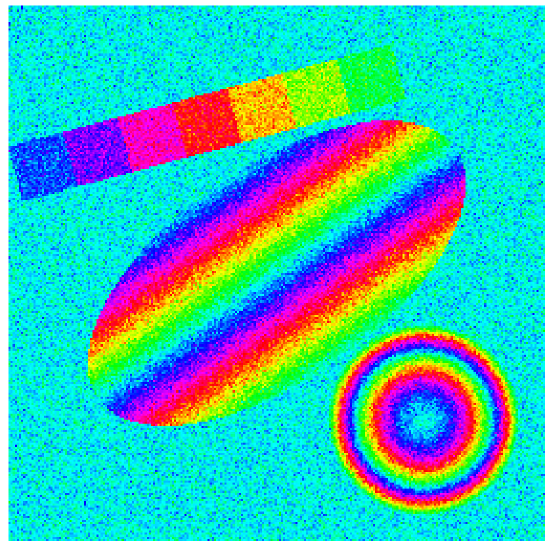
Applications

- Synthetic Manifold-Valued Data
- Real Manifold-Valued Data

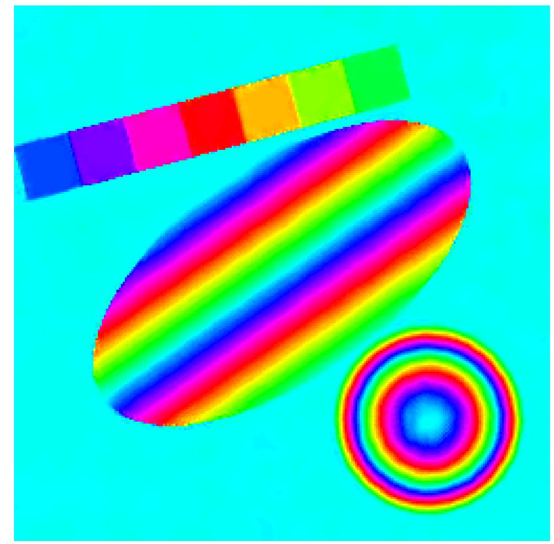
Denoising synthetic manifold-valued data: $\Omega \to \mathbb{S}^1$



Original phase data



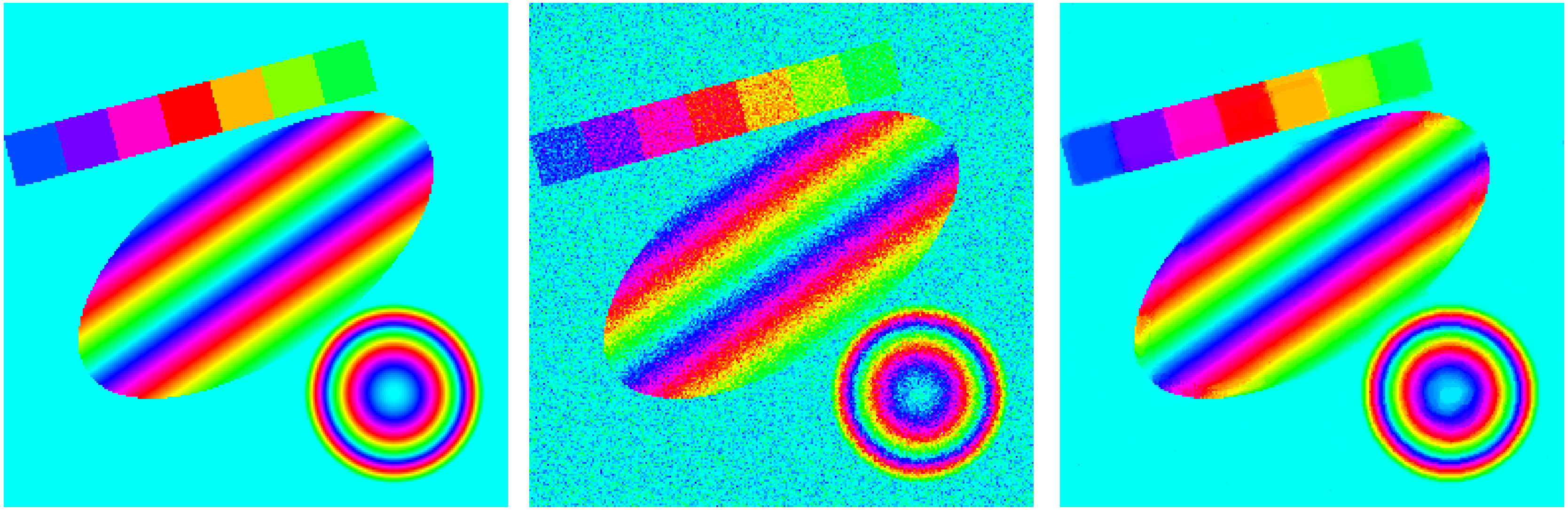
Noisy phase data
$\mathrm{MSE} = 0.0895$



NL-MSSE approach [8]
$\mathrm{MSE} = 0.00250$

[8] F. Laus, M. Nikolova, J. Persch, G. Steidl: *A Nonlocal Denoising Algorithm for Manifold- Valued Images Using Second Order Statistics*. SIAM Journal on Imaging Sciences 10(1), pp..416-448 (2017)

Denoising synthetic manifold-valued data: $\Omega \to \mathbb{S}^1$

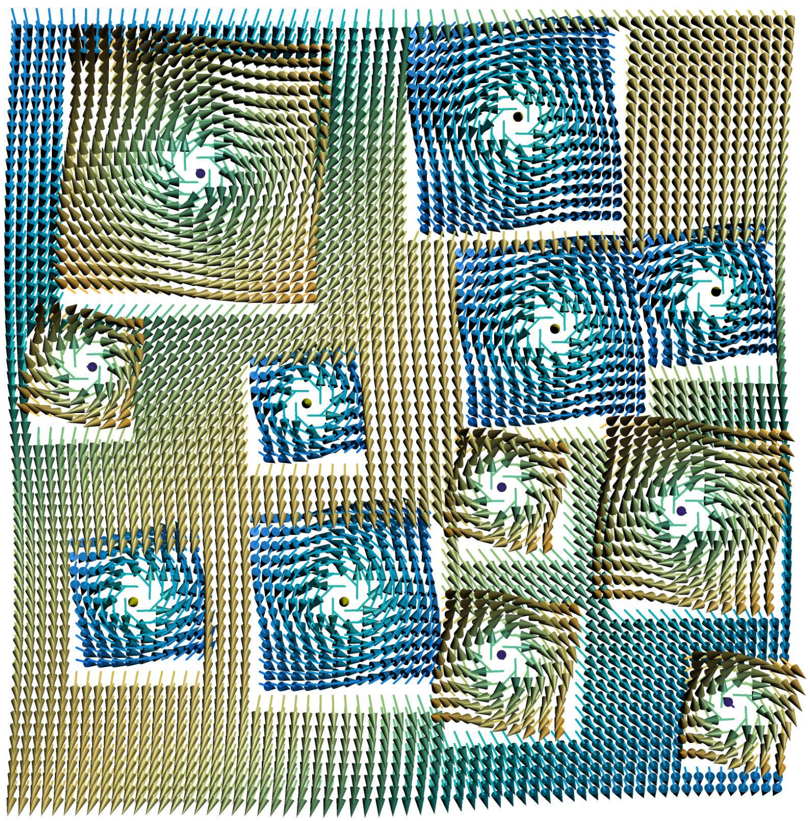


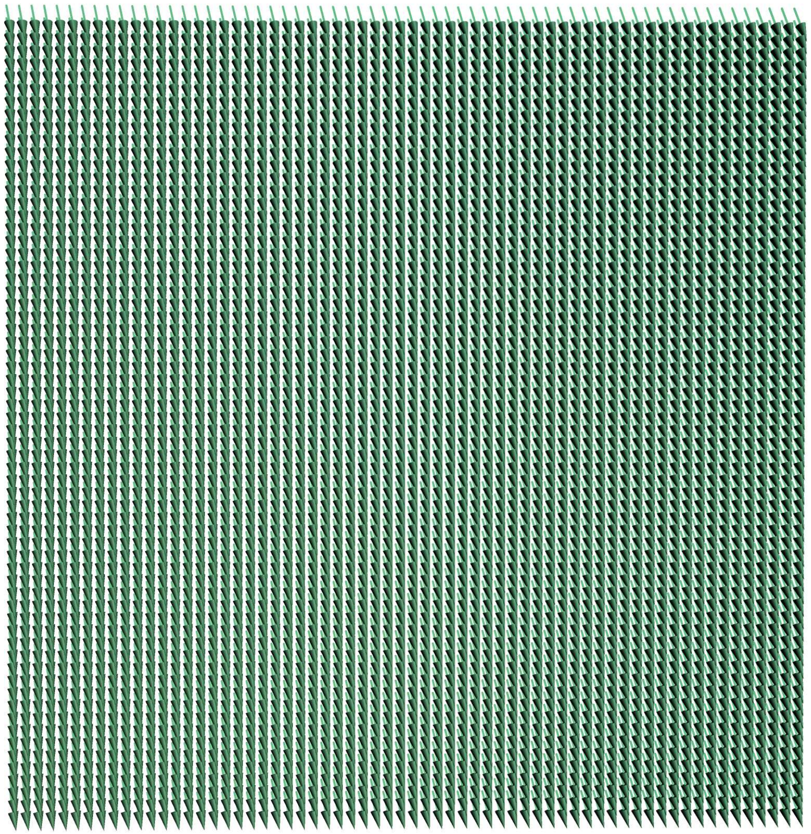
Original phase data

Noisy phase data
$\mathrm{MSE} = 0.0895$

Isotropic NL-TV approach
$\mathrm{MSE} = 0.00267$

Diffusion on synthetic manifold-valued data: $\Omega \to \mathbb{S}^2$

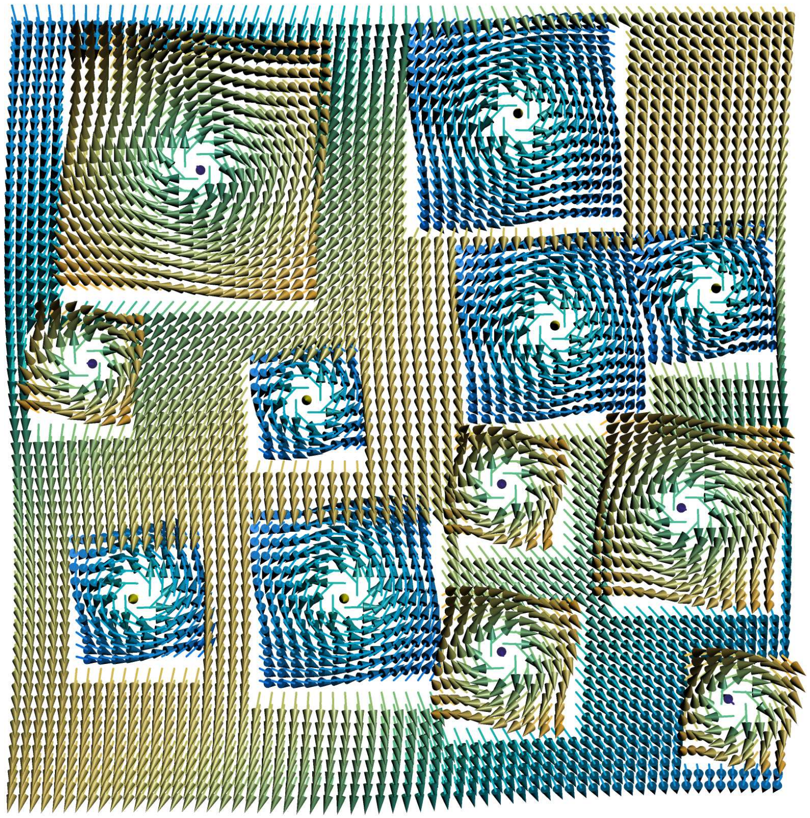


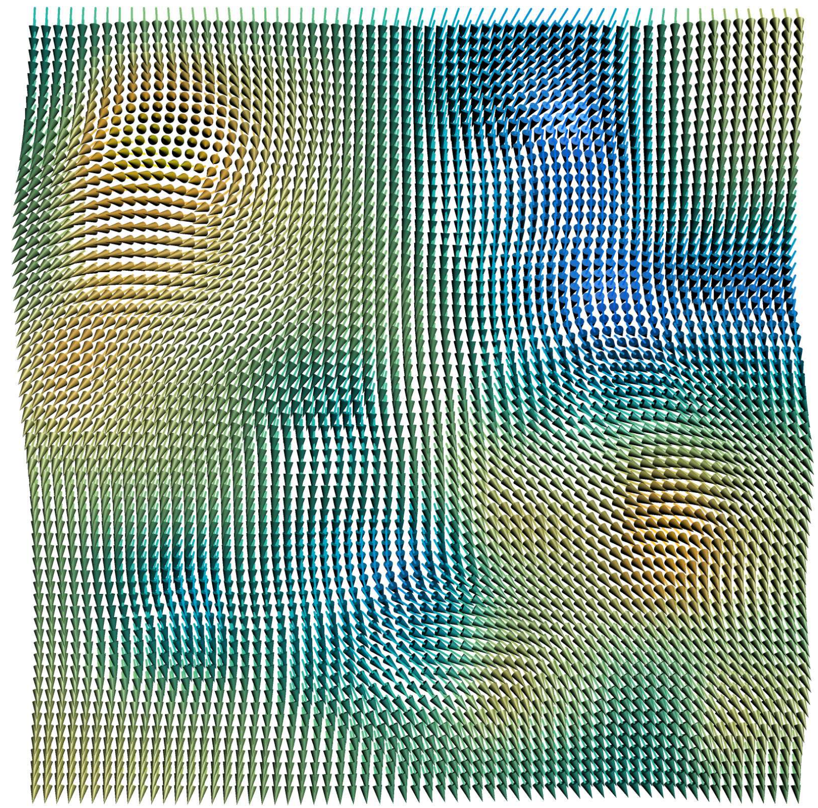


Heat flow for $p = 2$ and $\lambda = 0$

Video

Denoising on synthetic manifold-valued data: $\Omega \to \mathbb{S}^2$

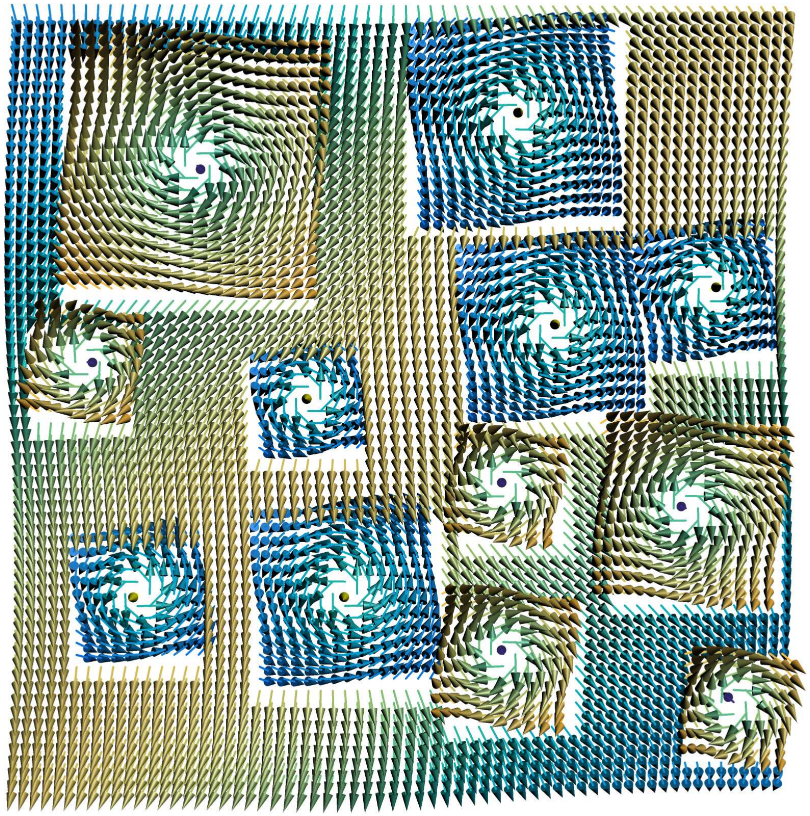


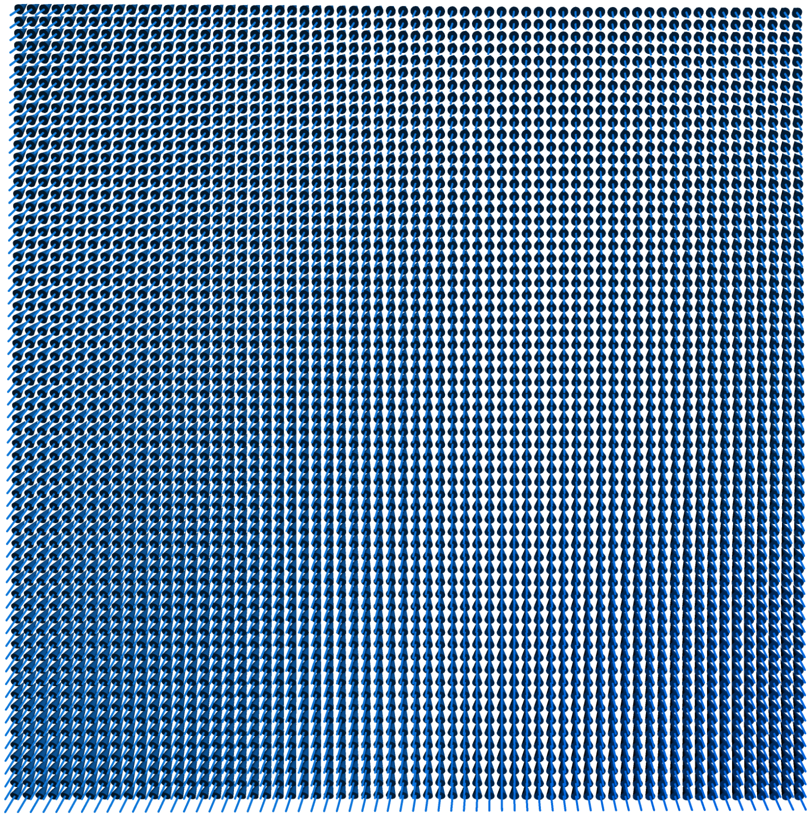


Tikhonov denoising for $p = 2$ and $\lambda = 0.01$

Video

Diffusion on synthetic manifold-valued data: $\Omega \to \mathbb{S}^2$

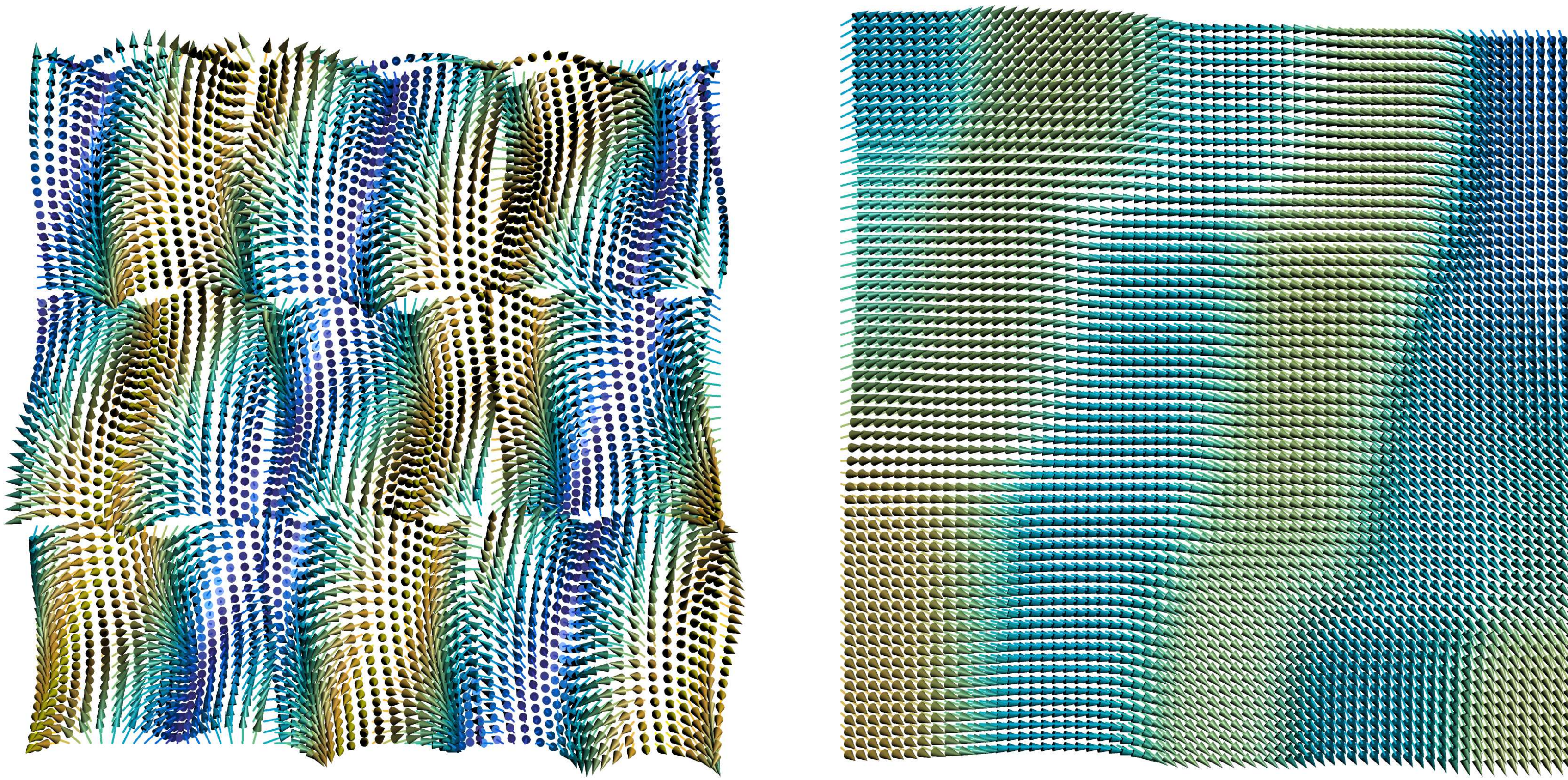




Anisotropic TV flow for $p = 1$ and $\lambda = 0$

Video

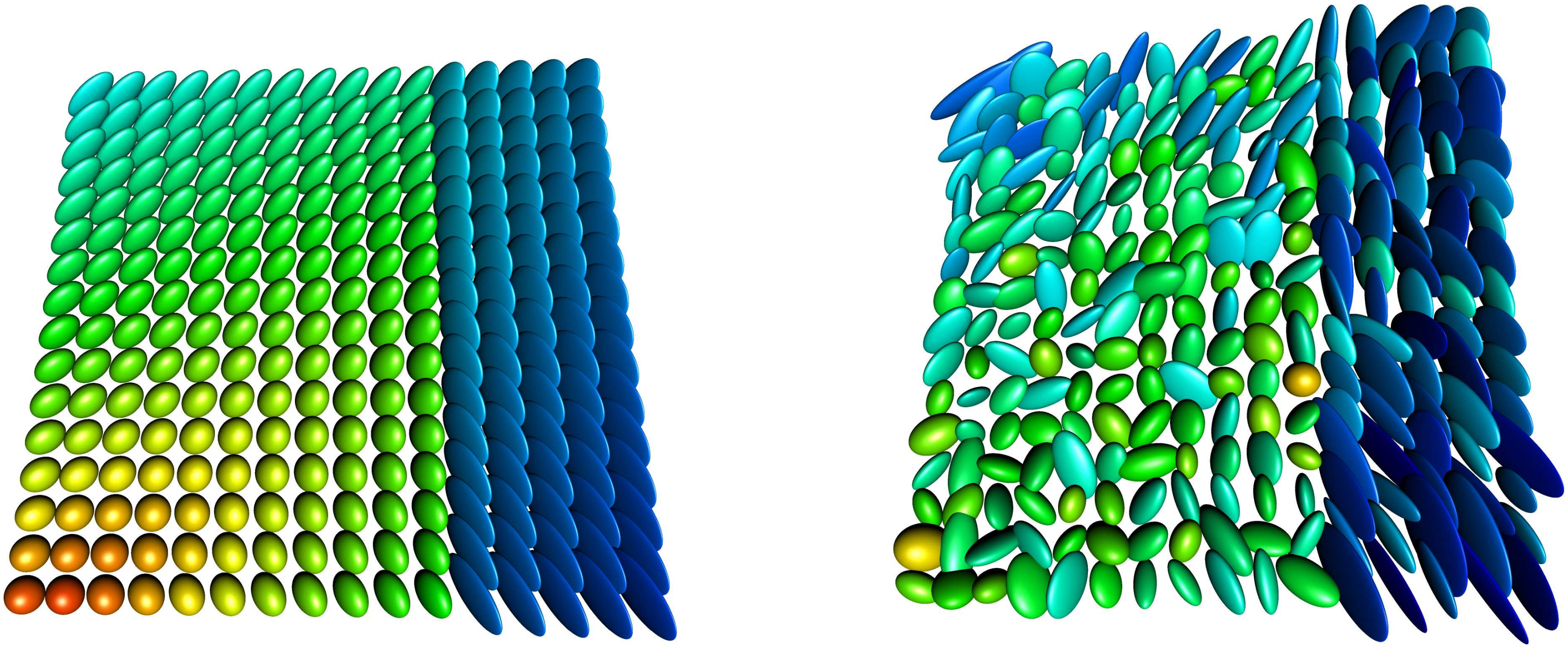
Denoising on synthetic manifold-valued data: $\Omega \to \mathbb{S}^2$



Anisotropic TV denoising for $p = 1$ and $\lambda = 0.001$

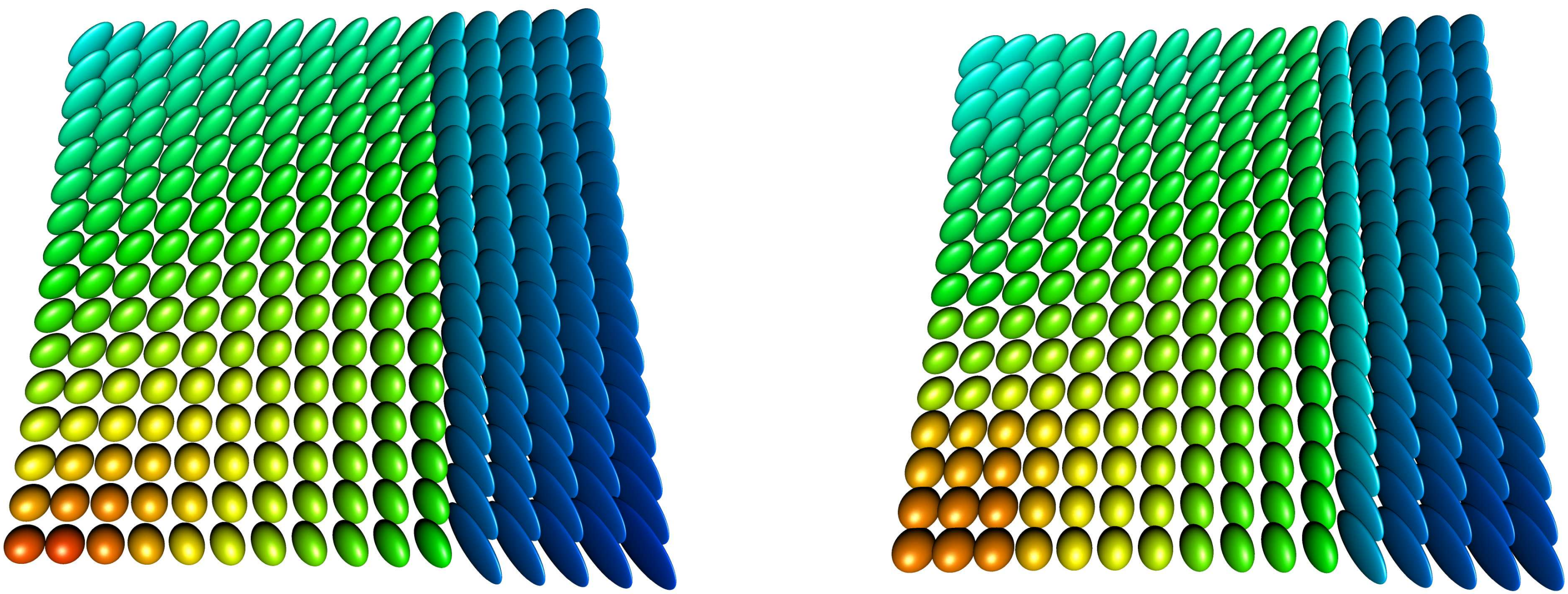
Video

Denoising on synthetic manifold-valued data: $\Omega \to \mathrm{SPD}(3)$



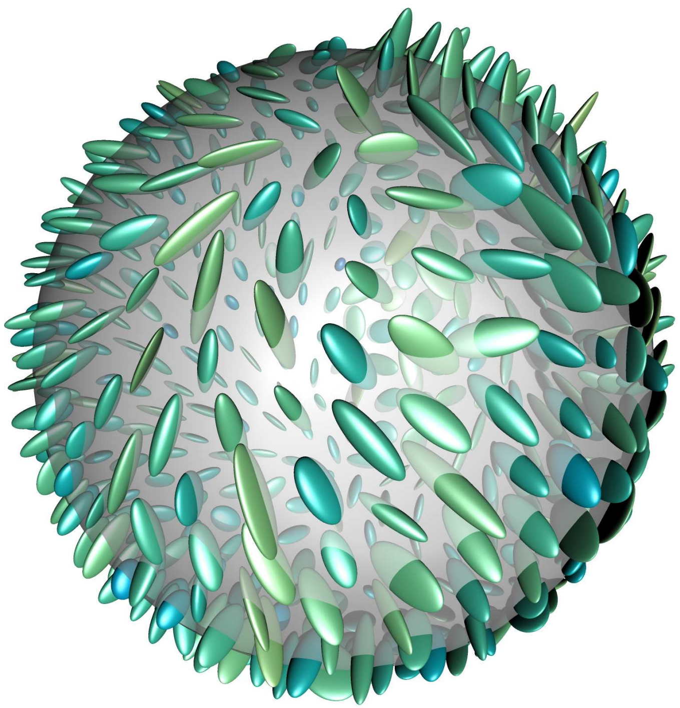
Anisotropic TV denoising result for $p = 1$ and $\lambda = 0.01$

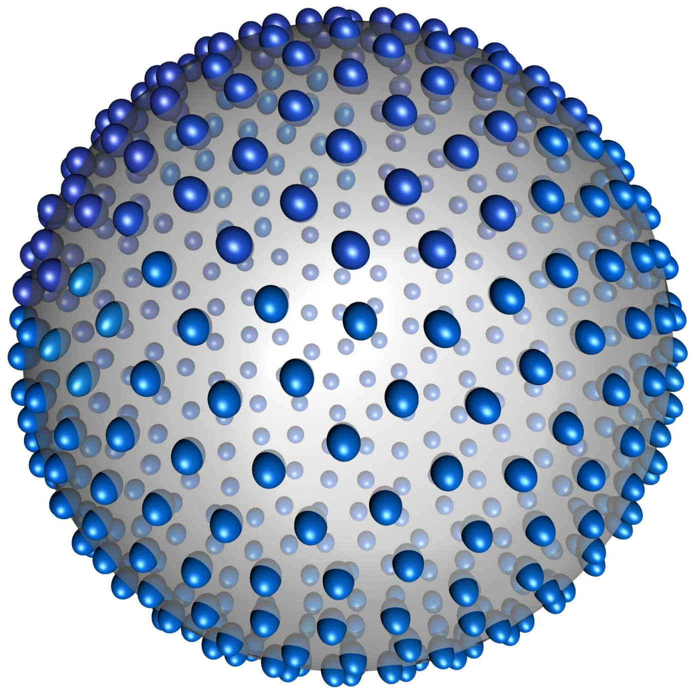
Denoising on synthetic manifold-valued data: $\Omega \to \mathrm{SPD}(3)$



Anisotropic TV denoising result for $p = 1$ and $\lambda = 0.01$

Denoising on synthetic manifold-valued data: $\mathbb{S}^2 \to \mathrm{SPD}(3)$





p-Laplace flow for $p = 1$ and $\lambda = 1$

Video

[9] M. Gräf: *Efficient algorithms for the computation of optimal quadrature points on Riemannian manifolds*. Ph.D. thesis at TU Chemnitz (2013)

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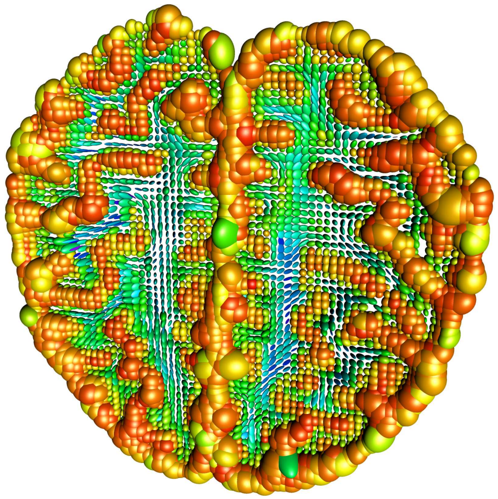
Denoising of real DT-MRI data: $\Omega \to \mathrm{SPD}(3)$

MRI system

- Diffusion tensor imaging (DTI) captures diffusion of water molecules

Denoising of real DT-MRI data: $\Omega \to \mathrm{SPD}(3)$

MRI system



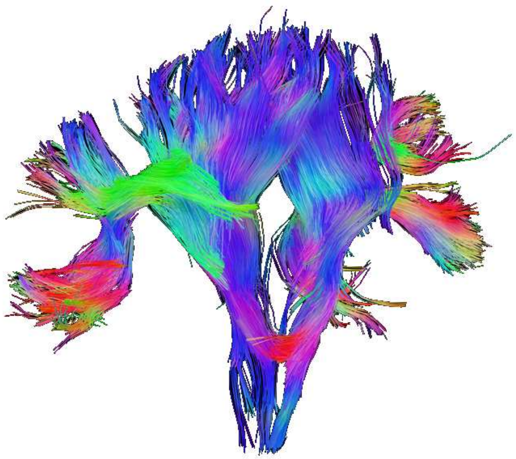
2D Slice from Camino dataset [10]

- Diffusion tensor imaging (DTI) captures diffusion of water molecules
- Diffusion tensors can be interpreted as manifold-valued data on $\mathrm{SPD}(3)$

[10] Cook et al.: *Camino: Open-Source Diffusion-MRI Reconstruction and Processing*. Proc. Intl. Soc. Mag. Reson. Med. 14 (2006) URL: http://hdl.handle.net/1926/38

Denoising of real DT-MRI data: $\Omega \to \mathrm{SPD}(3)$

MRI system



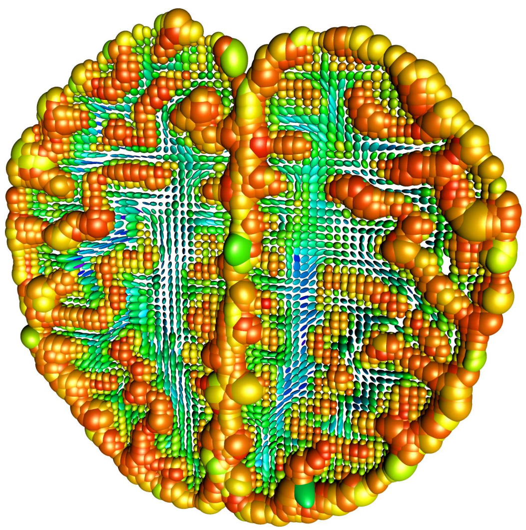
Reconstructed 3D fibres [11]

- Diffusion tensor imaging (DTI) captures diffusion of water molecules
- Diffusion tensors can be interpreted as manifold-valued data on $\mathrm{SPD}(3)$

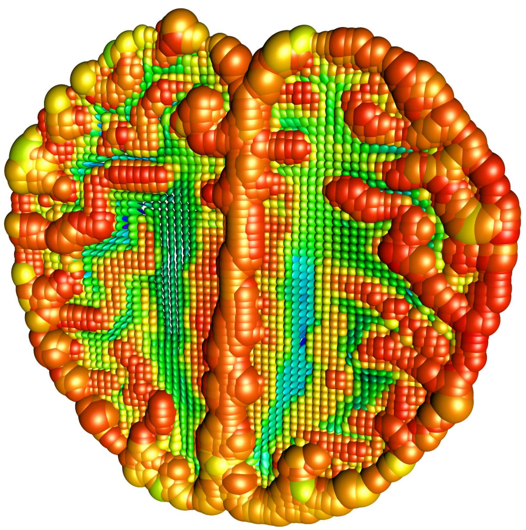
[10] Cook et al.: *Camino: Open-Source Diffusion-MRI Reconstruction and Processing*. Proc. Intl. Soc. Mag. Reson. Med. 14 (2006) URL: http://hdl.handle.net/1926/38

[11] http://www.humanconnectomeproject.org/

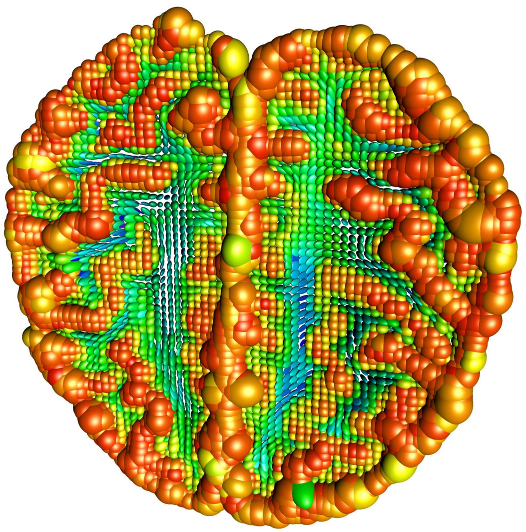
Denoising of real DT-MRI data: $\Omega \to \mathrm{SPD}(3)$



Original 2D slice

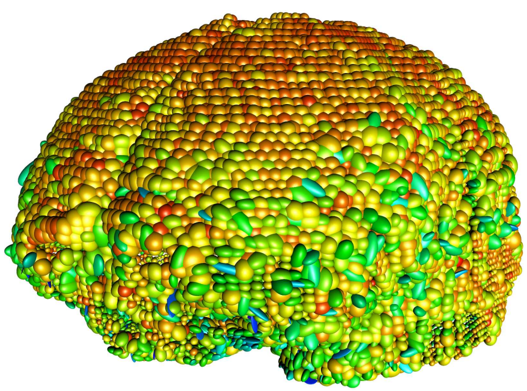


Anisotropic NL-TV for $\lambda = 10$



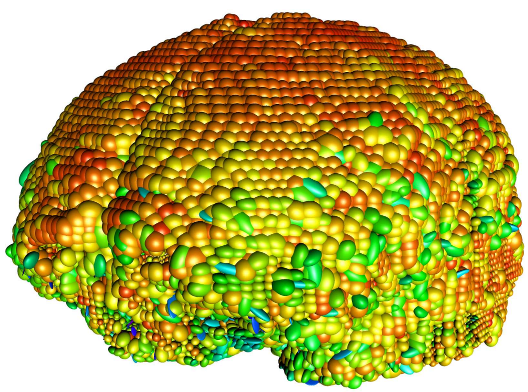
Isotropic NL-TV for $\lambda = 10$

Denoising of real DT-MRI data: $\Omega \to \mathrm{SPD}(3)$



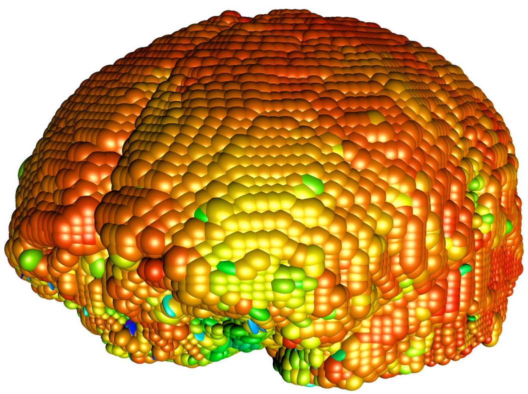
Original 3D surface data

▸ Video



Anisotropic TV for $\lambda = 50$

▸ Video



Anisotropic TV for $\lambda = 10$

▸ Video

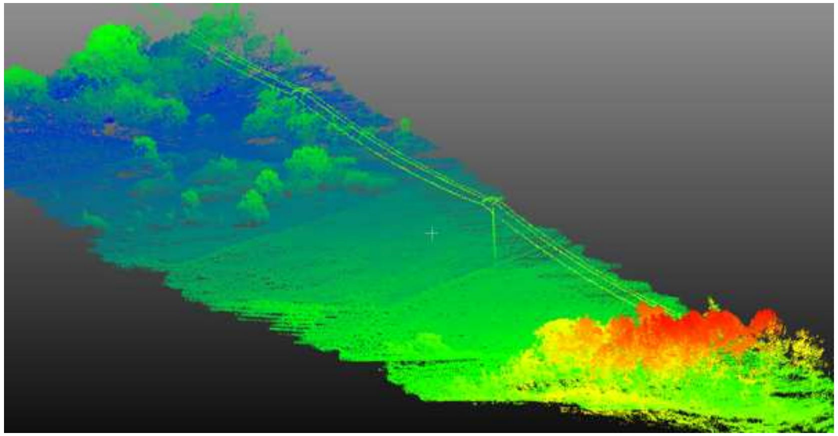
Denoising of real LiDAR imaging data: $\Omega \to \mathbb{S}^2$

Surveillance drone with LiDAR sensor

- Light detection and ranging (LiDAR) measures distance to objects

Denoising of real LiDAR imaging data: $\Omega \to \mathbb{S}^2$

Surveillance drone with LiDAR sensor

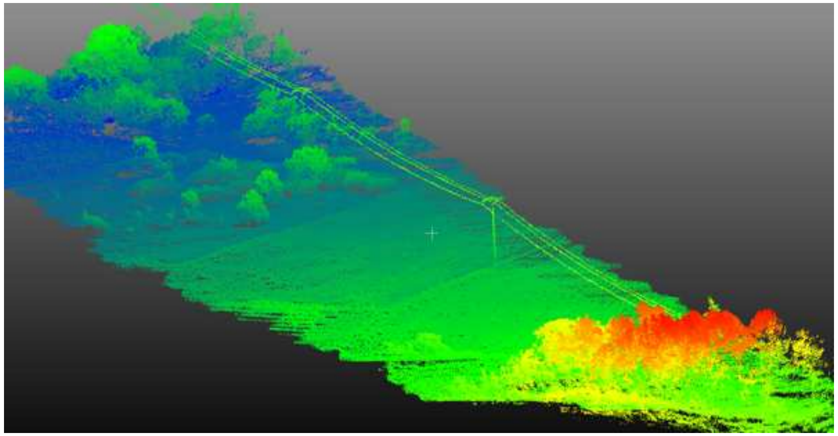


Acquired 3D point cloud of a landscape

- Light detection and ranging (LiDAR) measures distance to objects
- Surfaces are estimated from raw point cloud data

Denoising of real LiDAR imaging data: $\Omega \to \mathbb{S}^2$

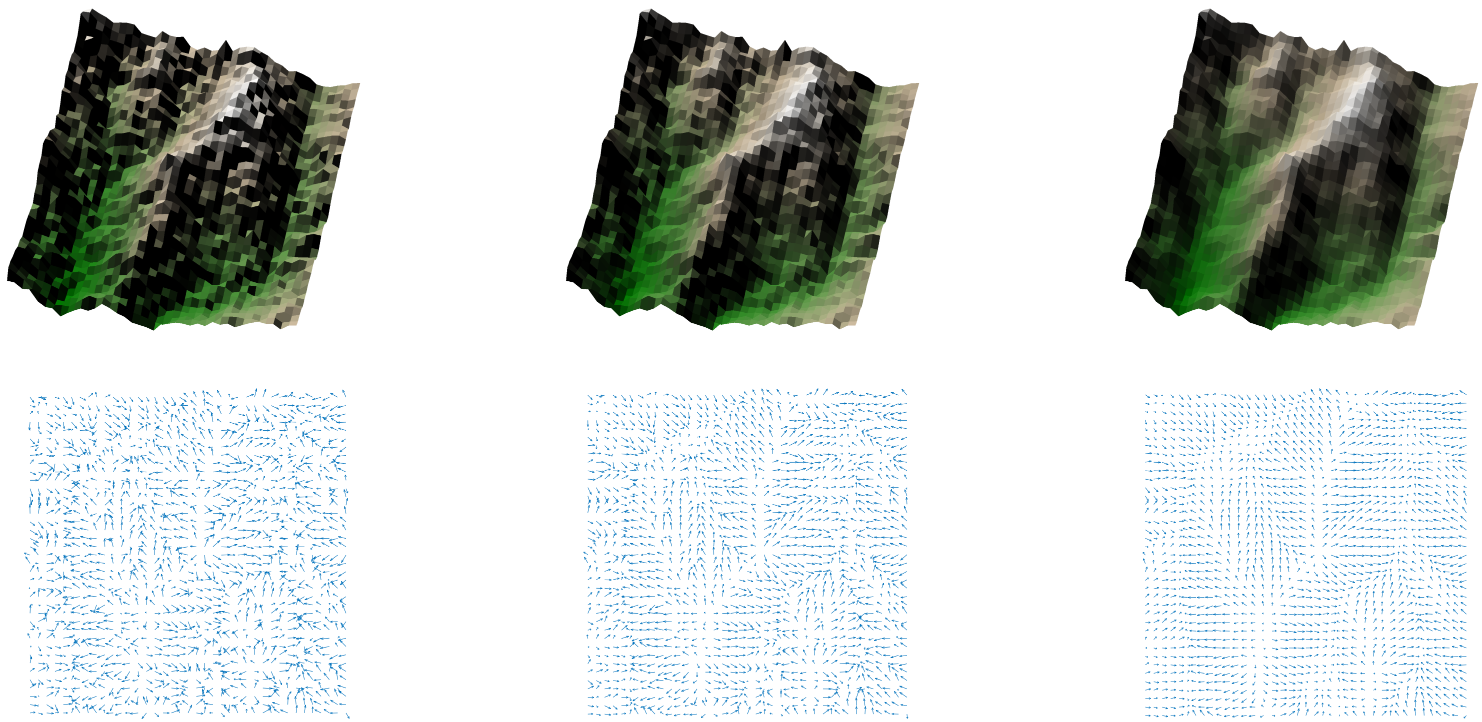
Surveillance drone with LiDAR sensor



Acquired 3D point cloud of a landscape

- Light detection and ranging (LiDAR) measures distance to objects
- Surfaces are estimated from raw point cloud data
- Surface normals can be interpreted as manifold-valued data on $\mathbb{S}^2$

Denoising of real LiDAR imaging data: $\Omega \to \mathbb{S}^2$



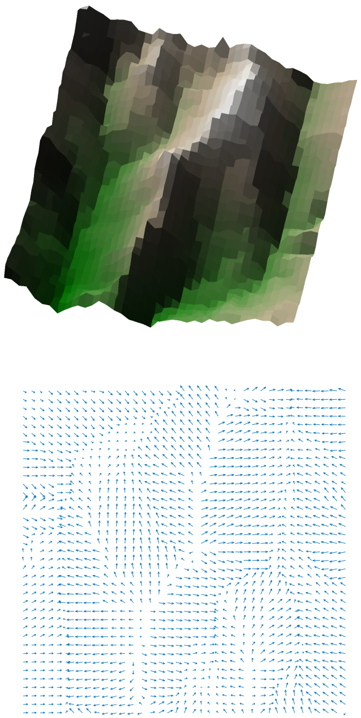
Original DEM data [19]

Reconstruction ($p = 2$, $\lambda = 5$)

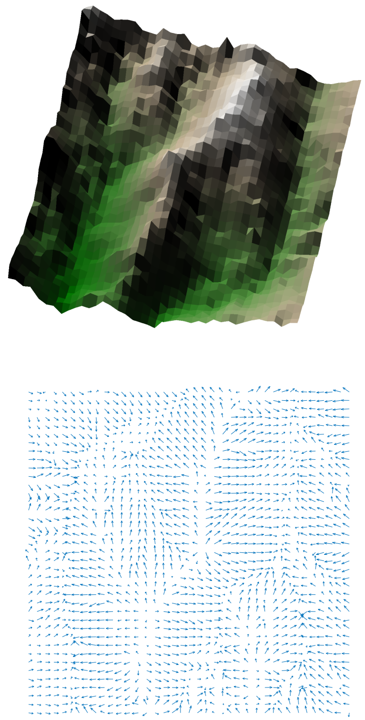
Reconstruction ($p = 2$, $\lambda = 0.5$)

[12] Gesch et al.: *The national map elevation*. Tech. rep. US Geological Survey (2009)

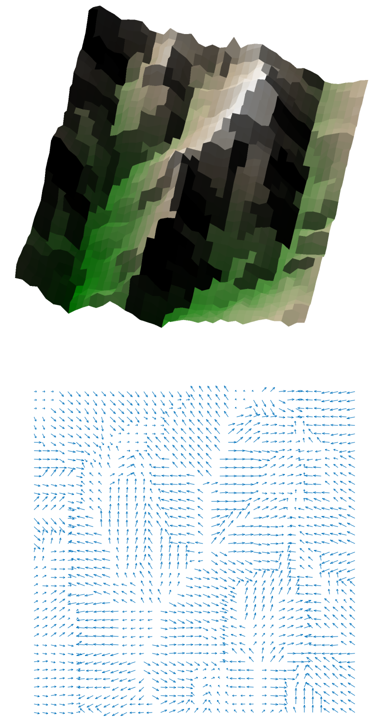
Denoising of real LiDAR imaging data: $\Omega \to \mathbb{S}^2$



Anisotropic TV recon.
($p = 1$, $\lambda = 2$)



Isotropic TV recon.
($p = 1$, $\lambda = 2$)



Sparse recon.
($p = 0.1$, $\lambda = 1$)

[12] Gesch et al.: *The national map elevation*. Tech. rep. US Geological Survey (2009)

- A. Elmoataz, M. Toutain, D. Tenbrinck: On the p-Laplacian and ∞-Laplacian on Graphs with Applications in Image and Data Processing. SIAM Journal on Imaging Sciences 8 (2015).
  HAL archive preprint: https://hal.archives-ouvertes.fr/hal-01247314
- D. Tenbrinck, F. Lozes, A. Elmoataz: Solving Minimal Surface Problems on Surfaces and Point Clouds. SSVM 2015
  Personal preprint: https://elmoatazbill.users.greyc.fr/point_cloud/2015_tenbrinck_ssvm.pdf
- R. Bergmann, D. Tenbrinck: Nonlocal Inpainting of Manifold-valued Data on Finite Weighted Graphs. *Geometric Science of Information – 3rd Conference on Geometric Science of Information* (2017)
  arXiv preprint: https://arxiv.org/abs/1704.06424
- R. Bergmann, D. Tenbrinck: A Graph Framework for Manifold-valued Data. *SIAM Journal on Imaging Sciences* 11 (2018).
  arXiv preprint: https://arxiv.org/abs/1702.05293
- R. Bergmann, R. Herzog, M. Silva Louzeiro, D. Tenbrinck, J. Vidal-Núñez: Fenchel-Duality for Convex Optimization and a Primal Dual Algorithm on Riemannian Manifolds (2020), submitted to: *Foundations of Computational Mathematics*
  arXiv preprint: https://arxiv.org/abs/1908.02022
- Open source **Matlab** software "Manifold-valued Image Restoration Toolbox (MVIRT)":
  https://github.com/kellertuer/MVIRT
- Open source **Julia** software "Optimization on Manifolds in Julia (Manopt.jl)":
  https://manoptjl.org

Thank you for your attention!

Any questions?

Acknowledgement:

DT acknowledges his prior funding by the ERC via grant EU FP7-ERC Consolidator Grant 615216 LifeInverse and the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 777826 (NoMADS).

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  arXiv preprint: https://arxiv.org/abs/1704.06424
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  arXiv preprint: https://arxiv.org/abs/1702.05293
- R. Bergmann, R. Herzog, M. Silva Louzeiro, D. Tenbrinck, J. Vidal-Núñez: Fenchel-Duality for Convex Optimization and a Primal Dual Algorithm on Riemannian Manifolds (2020), submitted to: *Foundations of Computational Mathematics*
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  https://manoptjl.org

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Any questions?

*“The contributions in this work are **manifold**...”* ☺

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Numerical realization

Observation:

We have *different options* to numerically approximate solutions to this PDE in the tangential space $T_{f(u)}\mathcal{M}$ for each $u \in \Omega$.

First method:

We consider the *parabolic equation*:

$$\frac{\partial f}{\partial t}(u,t) \;=\; \Delta_p^{\mathrm{a}} f(u,t) \;-\; \lambda \log_{f(u,t)} f_0(u,t) \quad \text{for all } (u,t) \in V \times [0,\infty)$$

For a stationary solution, i.e., $\frac{\partial f}{\partial t}(u,t) = 0$, we solve the original problem.

Note that for $\lambda = 0$ this equation covers two well-known *special cases*:

- $p = 1$: Total variation flow
- $p = 2$: Heat equation

Numerical realization

To solve the initial value problem

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} = \Delta_p^a f(u,t) \; - \; \lambda \log_{f(u,t)} f_0(u) \\ f(u,0) = f_0(u) \end{cases}$$

with Neumann boundary conditions we use an explicit Euler scheme:

$$\frac{\log_{f_n(u)} f_{n+1}(u)}{\Delta t} = \Delta_p^a f_n(u) \; - \; \lambda \log_{f_n(u)} f_0(u).$$

Thus, we have:

$$f_{n+1}(u) = \exp_{f_n(u)}(\Delta t \, (- \sum_{v \sim u} \sqrt{w(u,v)}^p d_{\mathcal{M}}(f_n(u), f_n(v))^{p-2} \log_{f_n(u)} f_n(v) - \lambda \log_{f_n(u)} f_0(u)))$$

Attention: For $p < 2$ this yields very strict CFL conditions on Δt.

Numerical realization

Second method:

We use a small trick by inserting two zero terms:

$$\begin{aligned} 0 &\overset{!}{=} \Delta^{\mathrm{a}}_{w,p} f(u) \;-\; \lambda \log_{f(u)} f_0(u) \\ &= -\sum_{v\sim u} \underbrace{\sqrt{w(u,v)}^{\,p}\, d_{\mathcal{M}}(f(u),f(v))^{p-2}}_{=:\gamma(u,v)} (\log_{f(u)} f(v) - \log_{f(u)} f(u)) \\ &\qquad - \lambda\,(\log_{f(u)} f_0(u) - \log_{f(u)} f(u)) \end{aligned}$$

We linearize the above problem by assuming that $\gamma(u,v)$ is known (by the last iteration) and hence we get a linear equation system $Af = b$.

Applying Jacobi's method we get the following relationship in $T_{f_n(u)}\mathcal{M}$:

$$\left(\lambda + \sum_{v\sim u} \gamma(u,v)\right) \log_{f_n(u)} f_{n+1}(u) = \sum_{v\sim u} \gamma(u,v) \log_{f_n(u)} f_n(v) \;+\; \lambda \log_{f_n(u)} f_0(u)$$

Numerical realization

We have the following relationship in $T_{f_n(u)}\mathcal{M}$:

$$\left(\lambda + \sum_{v\sim u} \gamma(u,v)\right) \log_{f_n(u)} f_{n+1}(u) \;=\; \sum_{v\sim u} \gamma(u,v) \log_{f_n(u)} f_n(v) \;+\; \lambda \log_{f_n(u)} f_0(u)$$

Hence, we get the following update formula:

$$f_{n+1}(u) \;=\; \exp_{f_n(u)}\left(\frac{\sum_{v\sim u} \gamma(u,v) \log_{f_n(u)} f_n(v) + \lambda \log_{f_n(u)} f_0(u)}{\lambda \,+\, \sum_{v\sim u} \gamma(u,v)} \right)$$

Observation:

For very small parameter λ (almost no data fidelity) this scheme is less robust as the explicit scheme.

Infinity Laplace operator

- let $\Omega \subset \mathbb{R}^d$ be a bounded, open set and $f \colon \Omega \to \mathbb{R}$ smooth
- the infinity Laplacian $\Delta_\infty f$ in $x \in \Omega$ can be defined $[13]$ as:

$$\Delta_\infty f(x) \; = \; \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_k} \frac{\partial^2 f}{\partial x_j x_k}(x).$$

- applications in image interpolation and inpainting $[14]$
- interesting connections to game theory, i.e., Tug-of-War games $[15]$

[13] M.G. Crandall, L.C. Evans, R.F. Gariepy: Optimal Lipschitz Extensions and the Infinity Laplacian. Calc. Var. Partial Differ. Equ 13 (2001)
[14] V. Caselles, J.M. Morel, C. Sbert: An Axiomatic Approach to Image Interpolation. Trans. Img. Proc. 7 (1998)
[15] Y. Peres, O. Schramm, S. Sheffield, D. Wilson: Tug-of-War and the Infinity Laplacian, J. Amer. Math. Soc. 22 (2009)

Min-max discretization

- simple approximation by min- and max-values in neighborhood [16]:

$$\Delta_\infty f(x) \;=\; \frac{1}{r^2}\left(\min_{y\in B_\epsilon(x)} f(y) + \max_{y\in B_\epsilon(x)} f(y) - 2f(x)\right) \;+\; \mathcal{O}(r^2).$$

- first graph-based variant proposed in [17]:

$$\begin{aligned}\Delta_\infty f(u) \;&=\; \|\nabla^+ f(u)\|_\infty - \|\nabla^- f(u)\|_\infty\\ &=\; \max_{v\sim u} |\min(\sqrt{w(u,v)}(f(v)-f(u)),0)|\\ &-\; \max_{v\sim u} |\max(\sqrt{w(u,v)}(f(v)-f(u)),0)|\end{aligned}$$

- **But:** operator restricted to real-valued vertex functions

[16] A.M. Oberman: A Convergent Difference Scheme for the Infinity Laplacian: Construction of Absolutely Minimizing Lipschitz Extensions. Math. Comp. 74 (2004)
[17] A. Elmoataz, X. Desquesnes, Z. Lakhdari, O. Lezoray: Nonlocal Infinity Laplacian Equation on Graphs with Applications in Image Processing and Machine Learning. Math. Comp. Sim. 102 (2014)

Connection to AML extensions

Observation: [18, 19]

Any (unique) viscosity solution f^* of the Dirichlet problem

$$\begin{cases} -\Delta_\infty f(x) \;=\; 0, & \text{for } x \in \Omega, \\ f(x) \;=\; \varphi(x), & \text{for } x \in \partial\Omega, \end{cases}$$

is an absolutely minimizing Lipschitz extension (AML) of φ, i.e.,

$$f^*(x) \;=\; g(x) \;\text{ for } x \in \partial\Sigma \;\Rightarrow\; ||Df^*||_{L^\infty(\Sigma)} \leq ||Dg||_{L_\infty(\Sigma)},$$

for every open, bounded subset $\Sigma \subset \Omega$ and every $g \in C(\overline{\Sigma})$.

[18] G. Aronsson: Extension of Functions Satisfying Lipschitz Conditions. Arkiv für Mate 6 (1967)
[19] R. Jensen: Uniqueness of Lipschitz Extensions Minimizing the sup-Norm of the Gradient. Arch. Rat. Mech. Anal. 123 (1993)

Constructing discrete Lipschitz extensions

- **Idea:** minimize locally the discrete Lipschitz constant [20]

$$\min_{f_0} L(f_0) \quad \text{with} \quad L(f_0) = \max_{x_j \sim x_0} \frac{|f_0 - f(x_j)|}{|x_0 - x_j|}$$

- this leads to a consistent scheme for solutions of $-\Delta_\infty f = 0$
- the infinity Laplace operator can be approximated by

$$\Delta_\infty f(x_0) = \frac{1}{|x_0 - x_j^*| + |x_0 - x_i^*|} \left(\frac{f(x_0) - f(x_j^*)}{|x_0 - x_j^*|} + \frac{f(x_0) - f(x_i^*)}{|x_0 - x_i^*|} \right)$$

for which the neighbors (x_i^*, x_j^*) are determined by:

$$(x_i, x_j) = \text{argmax}_{x_i, x_j \sim x_0} \frac{|u_i - u_j|}{|x_0 - x_i| + |x_0 - x_j|}$$

[20] A.M. Oberman: A Convergent Difference Scheme for the Infinity Laplacian: Construction of Absolutely Minimizing Lipschitz Extensions. Math. Comp. 74 (2004)

Graph infinity Laplacian for manifold-valued data

- we define the graph infinity Laplace operator for manifold valued data $\Delta_\infty f$ in a vertex $u \in V$ as

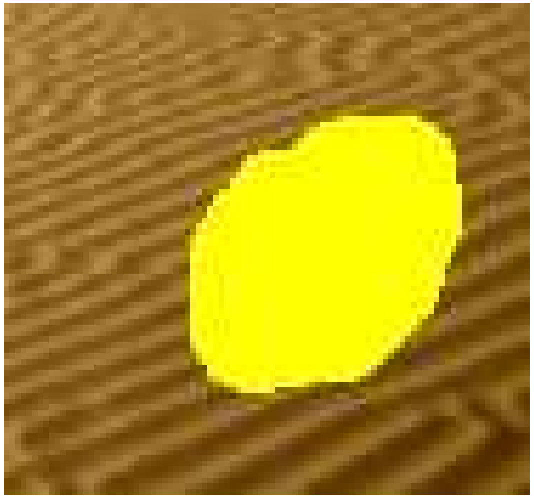
$$\Delta_\infty f(u) \;:=\; \frac{\sqrt{w(u,v_1^*)}\log_{f(u)} f(v_1^*) \;+\; \sqrt{w(u,v_2^*)}\log_{f(u)} f(v_2^*)}{\sqrt{w(u,v_1^*)} \;+\; \sqrt{w(u,v_2^*)}}$$

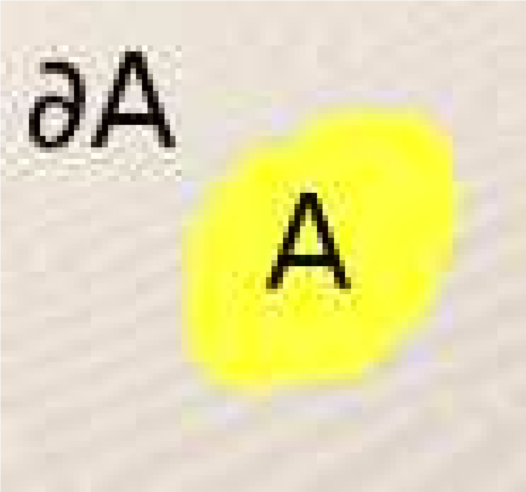
- $v_1^*, v_2^* \in \mathcal{N}(u)$ maximize the discrete Lipschitz constant in the local tangential plane $T_{f(u)}\mathcal{M}$ among all neighbors, i.e.,

$$(v_1^*, v_2^*) \;=\; \operatorname*{argmax}_{(v_1,v_2)\in\mathcal{N}^2(u)} \left\| \sqrt{w(u,v_1)}\log_{f(u)} f(v_1) - \sqrt{w(u,v_2)}\log_{f(u)} f(v_2) \right\|_{f(u)}$$

Interpolation problems

Goal: Inpaint $A \subset V$ using information in $\partial A = V/A$





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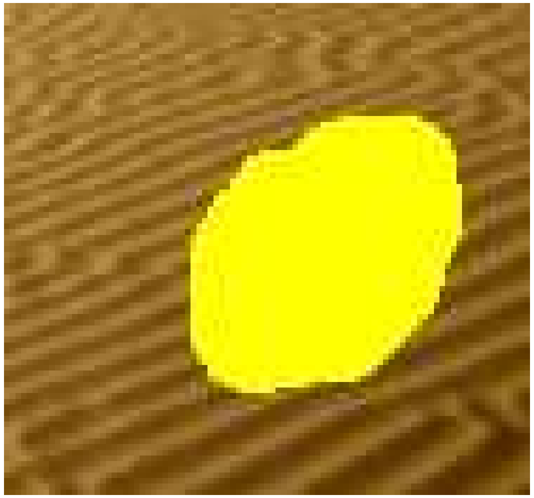

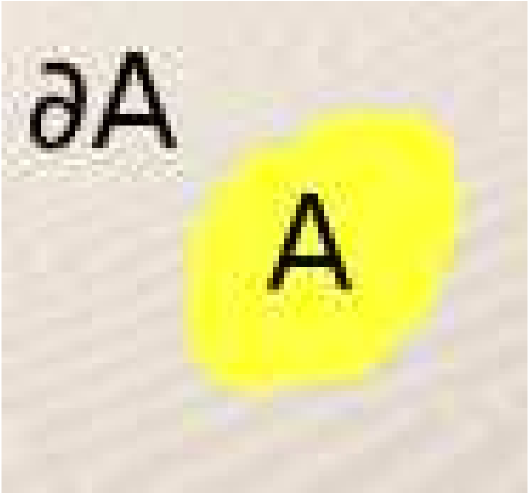

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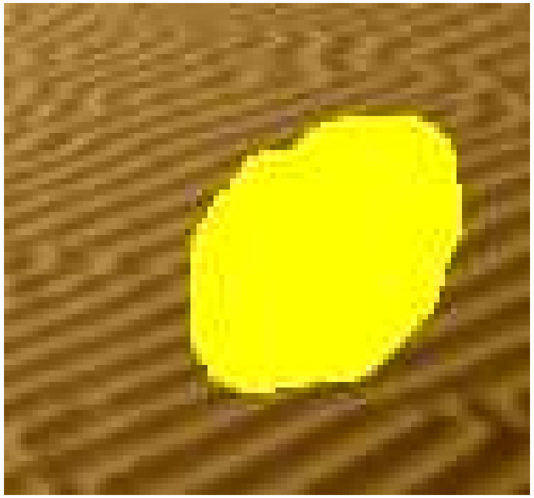

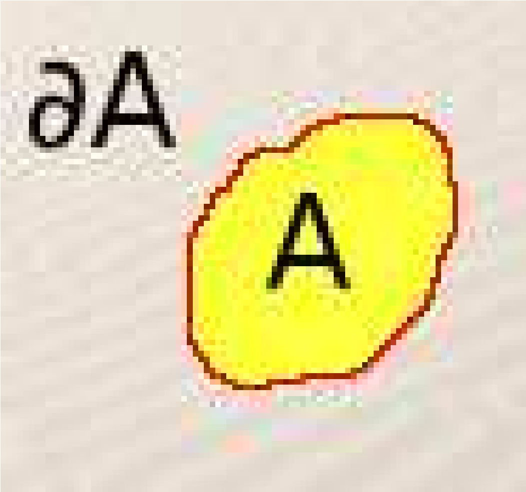

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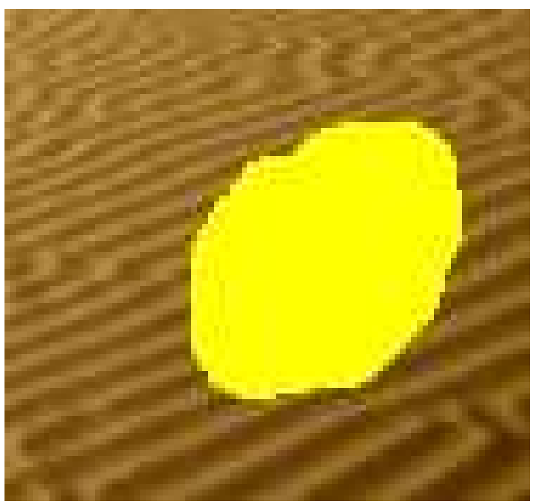

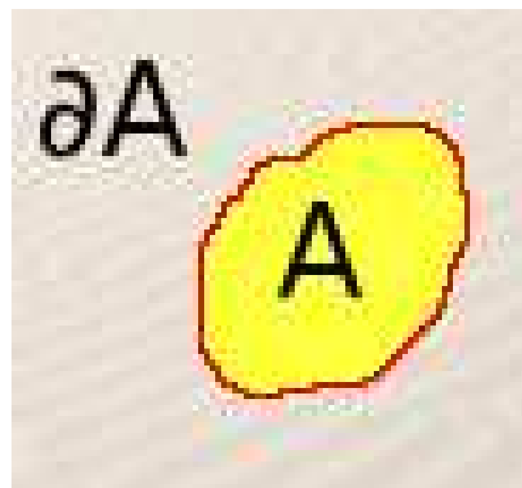

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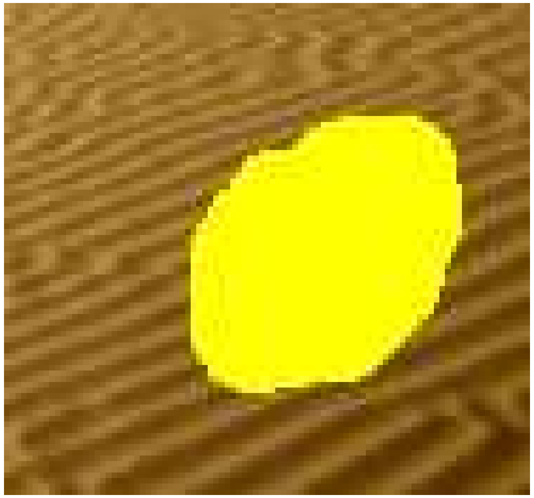

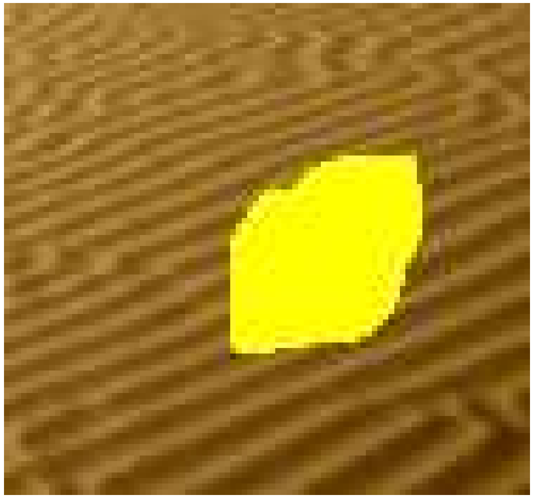

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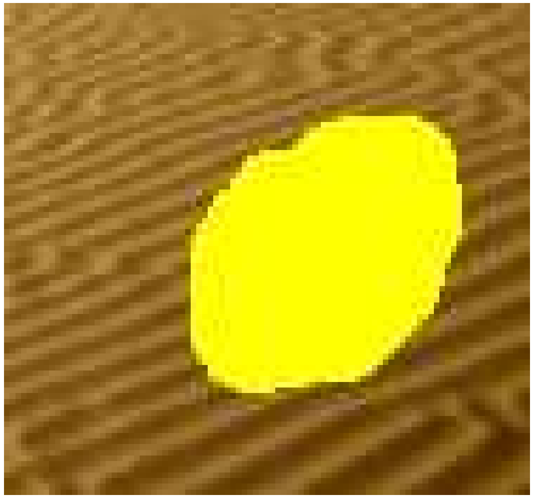

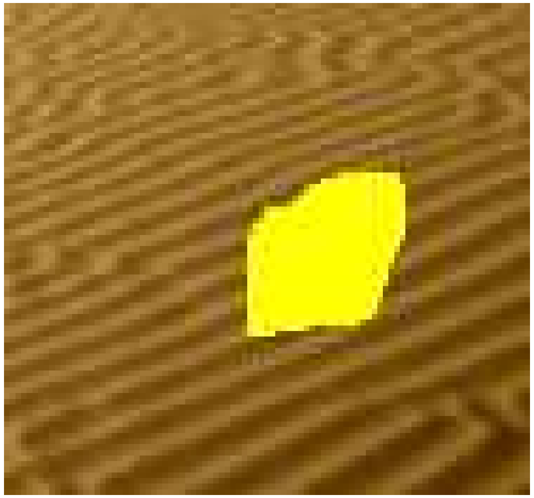

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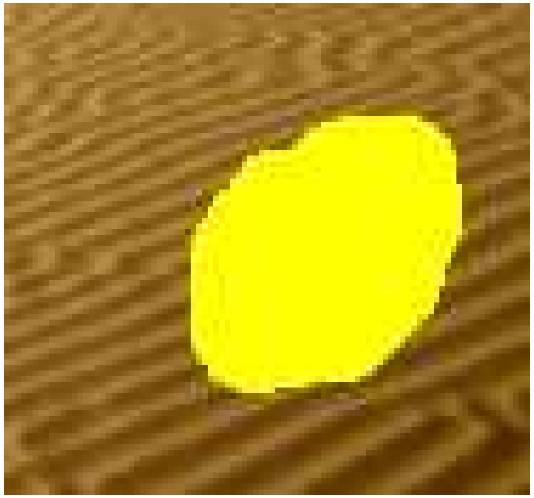

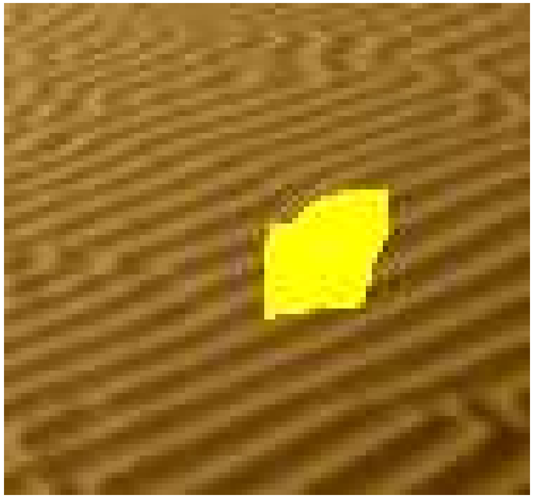

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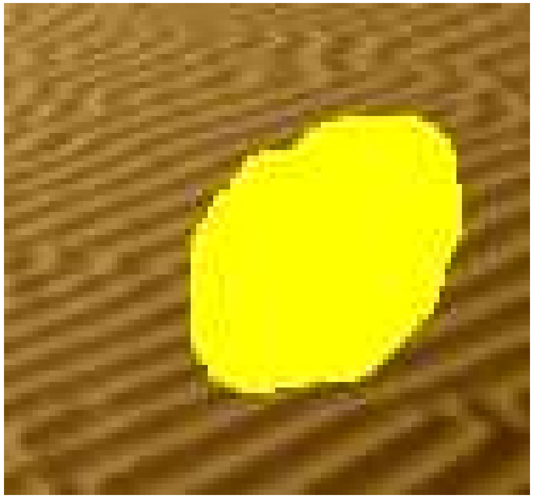

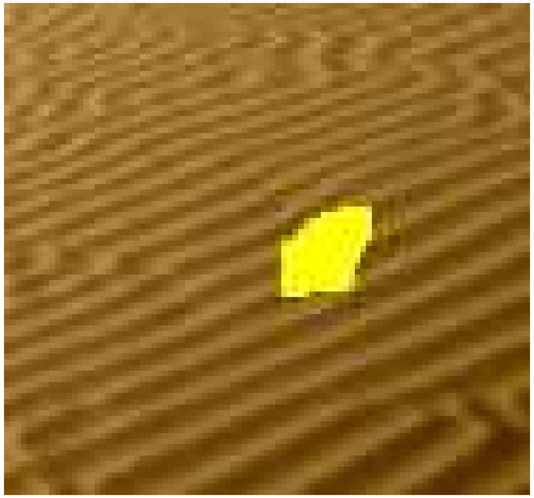

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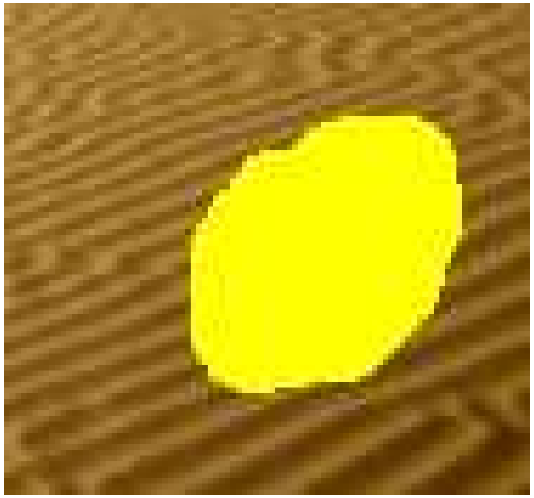

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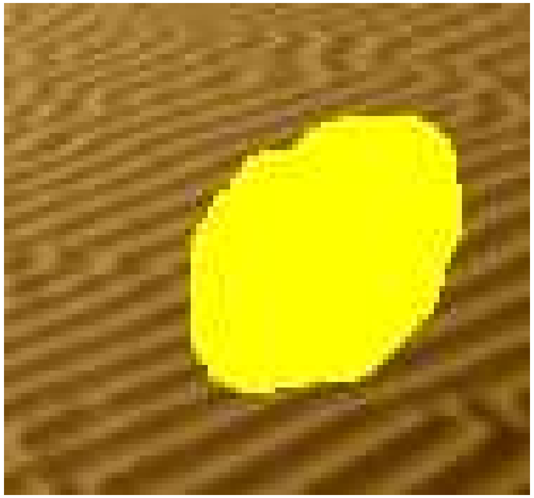

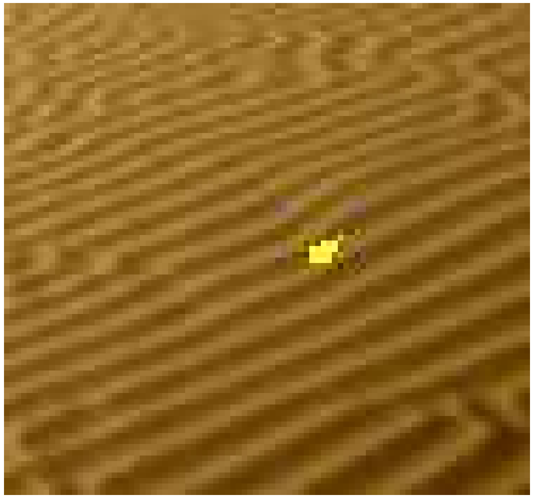

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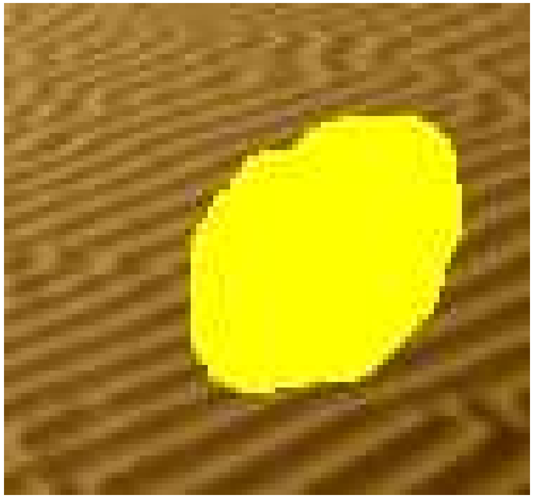

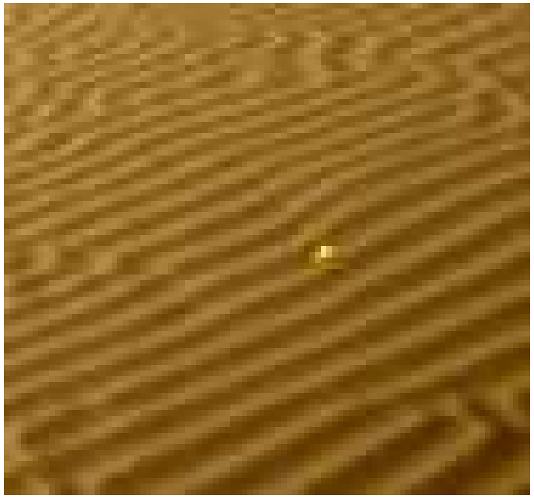

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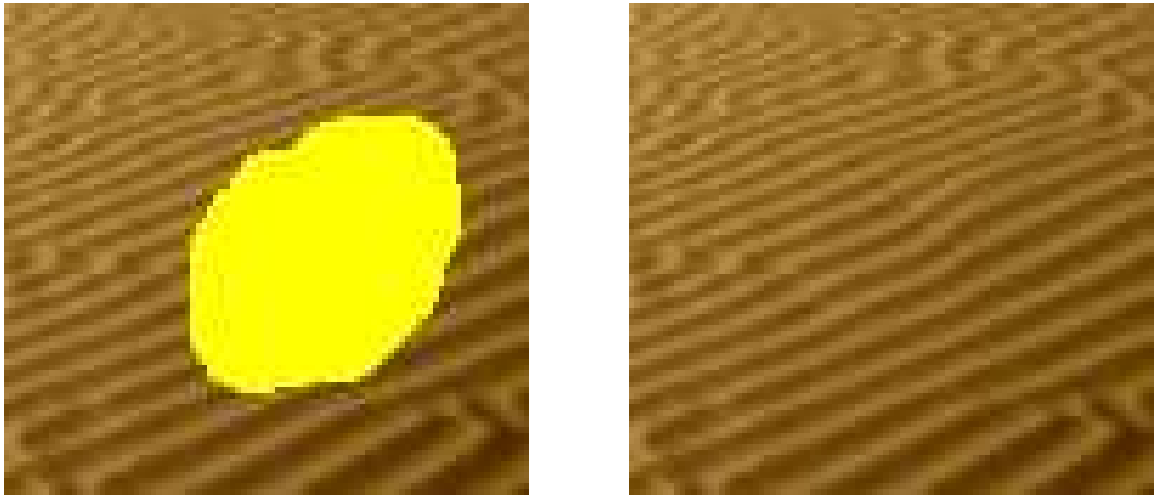
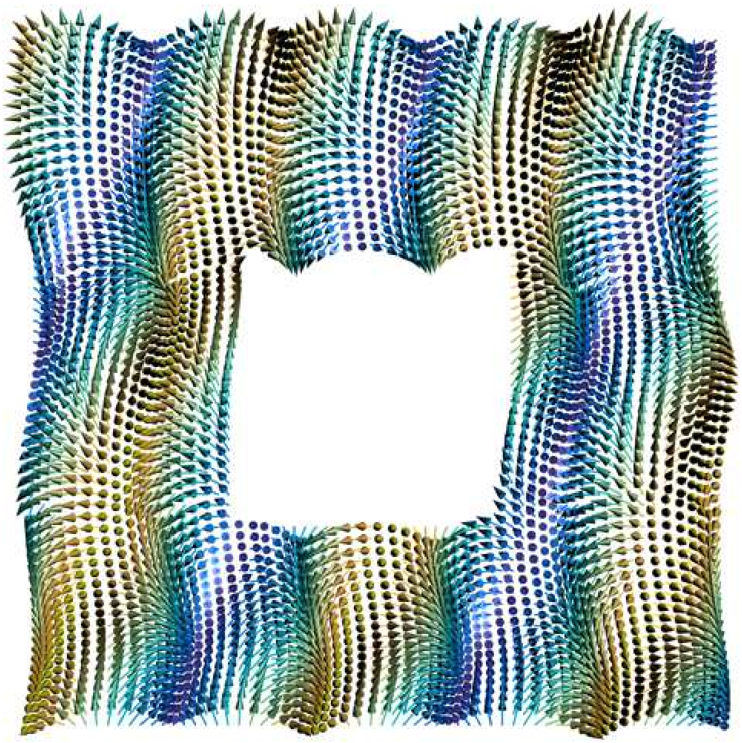
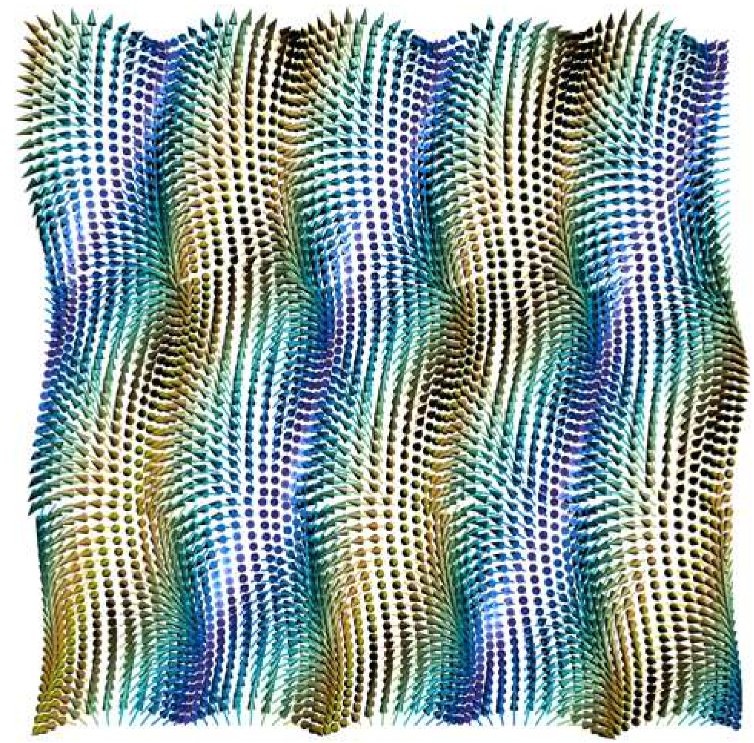


Inpainting of manifold-valued data $\Omega \to \mathcal{M}$

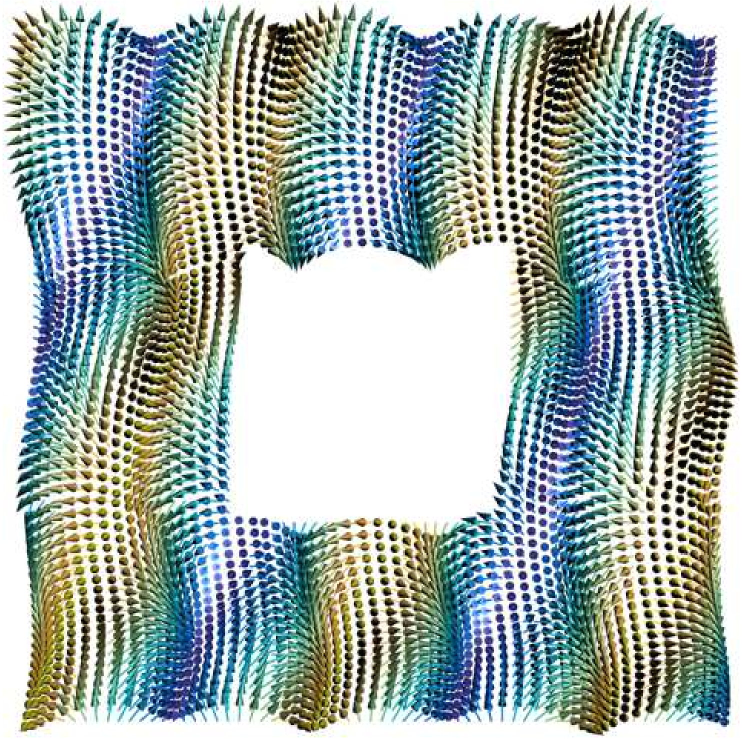


Manifold-valued data to be inpainted



Original data with values on $\mathcal{M} = \mathbb{S}^2$

Inpainting of manifold-valued data $\Omega \to \mathcal{M}$



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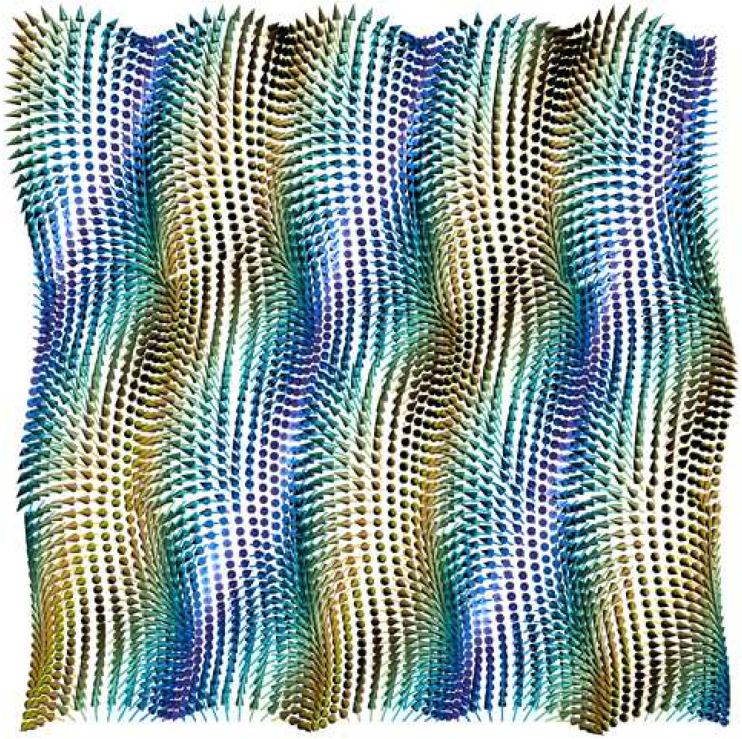

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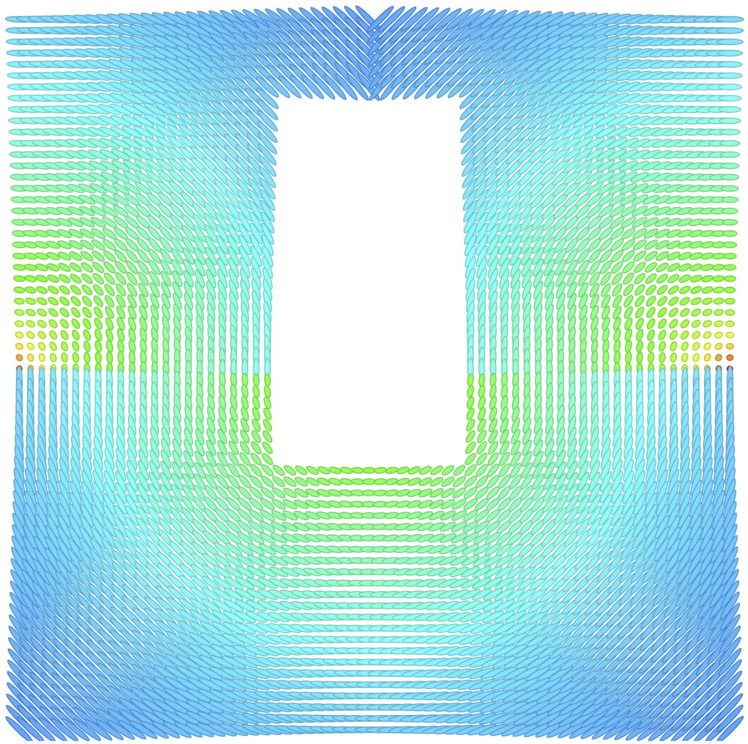

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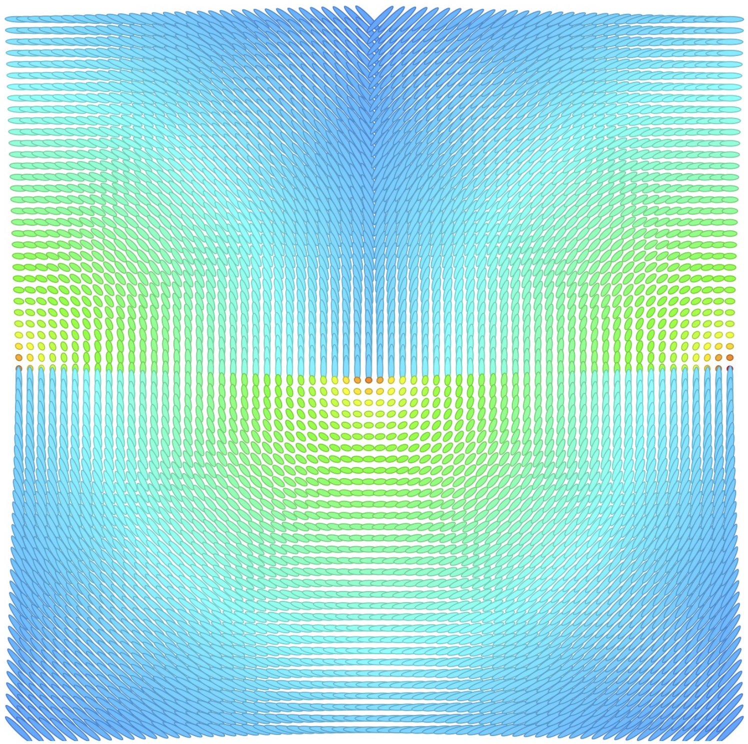
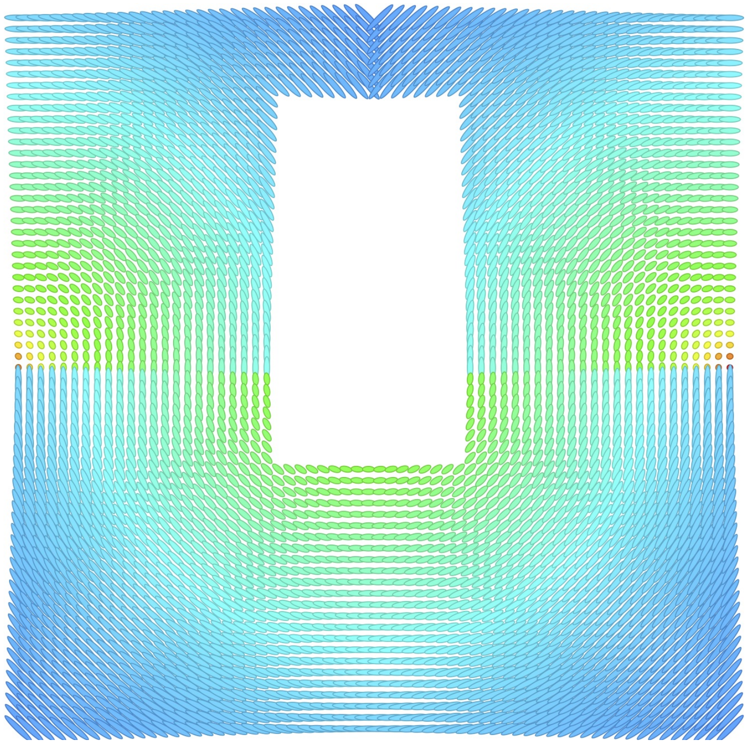


Original data with values on $\mathcal{M} = \mathrm{SPD}(2)$

Inpainting of manifold-valued data $\Omega \to \mathcal{M}$



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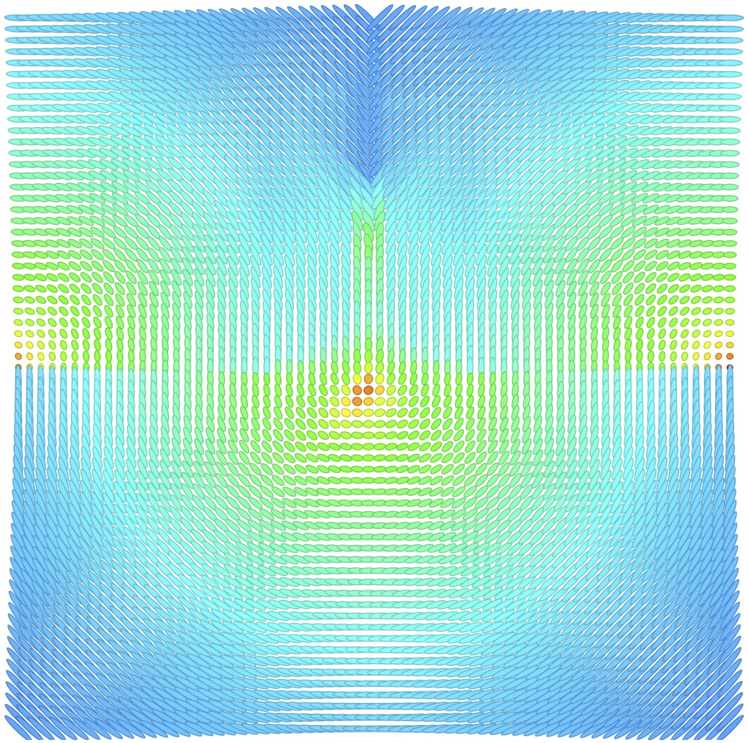

Inpainted data with values on $\mathcal{M} = \mathrm{SPD}(2)$